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BEAMS ON ELASTIC FOUNDATION

BEAMS ON ELASTIC FOUNDATION

THEORY WITH APPLICATIONS IN THE FIELDS OF CIVIL AND MECHANICAL ENGINEERING

BY

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PREFACE

The subject of this book is the analysis of elastically supported beams. The elastic support is provided here by a load-bearing medium, referred to as the "foundation," distributed continuously along the length of the beams. Such conditions of support can be found in a large variety of technical problems. In some of these problems the identity of the *beam* and the *foundation* can be easily established, as in the case of actual foundation structures or in the case of the railroad track. In other problems, however, which constitute perhaps the most fruitful field of application of this theory, the concept of beam and foundation is more of an abstract nature. Such conditions we find in networks of beams and in thin-walled tubes, shells, and domes, where the elastic foundation for the beam part is supplied by the resilience of the adjoining portions of a continuous elastic structure. Apart from the diversity of technical applications, there is a considerable variation possible in the fundamental subject itself. The flexural rigidity of the beam or the elasticity of the foundation may be a variable quantity; the axis of the beam may be straight or curved or the character of the applied loading may be axial, transverse, or torsional, in addition to a combination of end conditions to which any of these beams may be subjected. On the whole, however, all these problems are closely related through an affinity in their mathematical formulation. This renders the entire subject matter eminently suitable for a comprehensive treatment, which is the aim of the present volume.

In the course of this work much help was derived from the numerous publications on the subject, including several monographs in German and Russian, to which references are made in the footnotes. In attempting to form a comprehensive unit of all this material it has been found that many questions of interest to research men and practicing engineers have not yet been answered. This made it necessary to develop new solutions, to work out new cases of loadings, etc., the result of which is that a sizable portion of the material contained in this volume is of a kind that has not been published before. Among these new developments we may mention in particular the use of end-conditioning forces for producing beams of finite length under any combination of loading, the reduction of the problem of axially symmetrical deformation of conical and spherical shells to that of bending of beams on elastic foundation, and the introduction of the concept of foundation layers representing partial continuity in the material of the foundation. In addition to these a large number of new formulas for specific cases of loading and end conditions were worked out, together with illustrative examples, which appear interspersed throughout the text. Though the problems discussed are chiefly in the field of statics, the solutions developed in this connection may also be employed in other fields of mathematical physics, particularly in vibrations and acoustics.

There are two basic types of elastic foundations. The first type is characterized by the fact that the pressure in the foundation is proportional at every point to the deflection occurring at that point and is independent of pressures or deflections produced elsewhere in the foundation. Such a correlation between pressures and deflections implies a lack of continuity in the supporting medium, just as if it were made up of rows of closely spaced but independent elastic springs. The second type of foundation is furnished by an elastic solid which, in contrast to the first one, represents the case of complete continuity in the supporting medium. Though the first type is mathematically simpler, one should not regard it, as some investigators do, as an approximation or an "elementary" solution for the elastic solid foundation, because it has its own physical characteristics and significance. Foundations of the first type have by far the wider field of application in physical sciences, and most of the problems mentioned above can be reduced to elastic supporting conditions falling under this classification. For this reason the larger part of the book, nine chapters out of the ten, is devoted to problems arising in connection with such an essentially discontinuous type of foundation, and only the last chapter deals with cases in which the supporting body is an elastic continuum. Problems of continuity are introduced in the tenth chapter with a discussion of foundation layers which, with their varying and adjustable degree of continuity, form a useful transition between the two basic types of foundations mentioned above.

In the mathematical notation of this text a minor departure was made from existing practice in that capital initial letters are used in the otherwise customary notations for hyperbolic functions. The need for this arose from the fact that, owing to the nature of the subject, solutions often appeared in lengthy and sometimes perplexing combinations of trigonometric and hyperbolic functions. Thus it became highly desirable to accentuate the difference in notation between these two types of functions, and the use of a capital initial for the latter type was found to be a simple and effective way to achieve the purpose.

The first manuscript for this book was prepared in 1936–37 during the tenure of a Horace H. Rackham Postdoctorate Fellowship at the University of Michigan. Since then the material has been revised several times and enlarged until it has assumed its present form. The author takes this opportunity to express his deep appreciation and gratitude to the University of Michigan for granting the generous fellowship which made this undertaking possible and for supporting the publication of the ensuing results. During the work much encouragement and benefit were derived from personal contacts with Professor Stephen P. Timoshenko, who first aroused the author's interest in this subject and who proved to be a constant source of inspiration. It is also a pleasure to acknowledge the valuable assistance received from Professor Edward L. Eriksen, Dr. Merhyle F. Spotts, and Dr. Stewart Way. The author is greatly indebted to Dr. Eugene S. McCartney, editor of the University of Michigan Press, for his care in steering the publication through the press under wartime conditions and for the many constructive suggestions that both he and Miss Grace Potter, former assistant editor, have contributed.

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CONTENTS

CHAPTER X

TABLES

CHAPTER I

GENERAL SOLUTION OF THE ELASTIC LINE

In the major part of this work the analysis of bending of beams on an elastic foundation is developed on the assumption that the reaction forces of the foundation are proportional at every point to the deflection of the beam at that point. This assumption was introduced first by E. Winkler* in 1867 and formed the basis of H. Zimmermann's classical work† on the analysis of the railroad track, published in 1888. Though the early investigators thought chiefly of soil as the supporting medium, it was later found that there are other fields where the conditions of Winkler's assumption are much more rigorously satisfied. Two such fields of application were discovered to be of particular importance, and they are discussed in detail in the course of this book. One of these is concerned with networks of beams, which are characteristic in the construction of floor systems for ships, buildings, and bridges; the other deals with thin shells of revolution and includes such subjects as pressure vessels, boilers, and containers, as well as large-span modern reinforced concrete halls and domes. While the theory of beams on elastic foundation holds rigidly for most of the problems mentioned above, its application to soil foundations should be regarded only as a practical approximation. The physical properties of soils are obviously of a much more complicated nature than that which could be accurately represented by such a simple mathematical relationship as the one assumed by Winkler. There are, however, some important points which can be brought up in supporting the application of this theory to soil foundations. Under certain conditions the elasticity of soil is undeniable; it can propagate sound waves, for instance. Also, the second, and most debated, part of Winkler's assumption, that the foundation deforms only along the portion directly under loading, has, since A. Föppl's classical experiment,‡ often been found to be true for a large variety of soils. If we take these things into consideration, there is reason to believe that the Winkler theory, in spite of its simplicity, may often more accurately represent the actual conditions existing in soil foundations than do some of the more complicated analyses advanced in recent years and discussed in the last chapter of this book, where the foundation is regarded as a continuous isotropic elastic body. Which one of these theories to apply, and how much continuity in the supporting medium to assume, can be decided, however, in a given case only by physical testing of the material of the foundation under consideration.

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* *Die Lehre von der Elastizität und Festigkeit* (Prag, 1867), p. 182.

† *Die Berechnung des Eisenbahnoberbaues* (Berlin, 1888; 2d ed., Berlin, 1930).

‡ A. Föppl, *Vorlesungen über technische Mechanik* (9th ed.; Leipzig, 1922), III, 258.

1. *The Differential Equation of the Elastic Line*

Consider a straight beam supported along its entire length by an elastic medium and subjected to vertical forces acting in the principal plane of the symmetrical cross section (Fig. 1). Because of this action the beam will deflect, producing continuously distributed reaction forces in the supporting medium. Regarding these reaction forces we make the fundamental assumption that their intensity p at any point is proportional to the deflection of the beam y at that point: $p = ky$ The reaction forces will be assumed to be acting vertically and opposing the deflection of the beam. Hence where the deflection is directed downward (positive) there will be a compression in the supporting medium, but, on the other hand, where the deflection happens to be negative, tension will be produced; for the present we suppose the supporting medium to be able to take up such tensile forces.

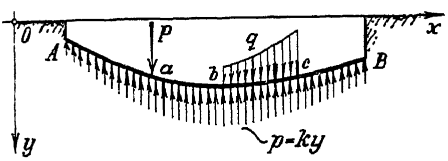

FIG. 1

The assumption $p = ky$ implies the statement that the supporting medium is elastic; in other words, that its material follows Hooke's law. Its elasticity, therefore, can be characterized by the force which, distributed over a unit area, will cause a deflection equal to unity. This constant of the supporting medium, k_0 lbs./in.^{3}, is called the *modulus of the foundation.*

Assume that the beam under consideration has a uniform cross section and that b is its constant width, which is supported on the foundation. A unit deflection of this beam will cause reaction bk_0 in the foundation; consequently, at a point where the deflection is y the intensity of distributed reaction (per unit length of the beam) will be

$$p \text{ lbs./in.} = bk_0y. \tag{a}$$

For the sake of brevity we shall use the symbol k lbs./in.^{2} for b in. $\times$ k_0 lbs./in.^{3} in the following derivations, but it is to be remembered that this k includes the effect of the width of the beam and will be numerically equal to k_0 only if we deal with a beam of unit width.

While the loaded beam deflects, it is possible that besides the vertical reactions there may also be some horizontal (frictional) forces originating along the surface where the beam is in contact with the foundation. The influence of such horizontal forces on the deflection line will be shown in a later chapter; for the present their (possibly small) effect will not be considered, and the reaction forces on the foundation will be assumed to be vertical at every cross section.

Let us take an infinitely small element enclosed between two vertical* cross

* By this we assume that the slope is so small compared to unity that cross sections (normal to the elastic line) can be replaced by vertical sections. Such approximation cannot be used when investigating the effect of axial forces on the deflection of the beam (Chapter VI).

sections a distance dx apart on the beam under consideration. Assume that this element was taken from a portion where the beam was acted upon by a distributed loading q lbs./in. The forces exerted on such an element are shown in Figure 2. The upward-acting shearing force, Q, to the left of the cross section is considered positive, as is the corresponding bending moment, M, which is a clockwise moment acting from the left on the element (the moment of a positive Q). These positive directions for Q and M will be kept in all later derivations. Considering the equilibrium of the element in Figure 2, we find that the summation of the vertical forces gives

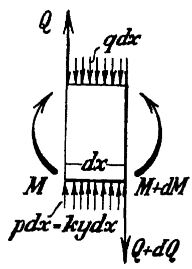

FIG. 2

$$Q - (Q + dQ) + ky\,dx - q\,dx = 0,$$

whence

$$\frac{dQ}{dx} = ky - q. \tag{b}$$

Making use of the relation $Q = dM/dx$, we can write

$$\frac{dQ}{dx} = \frac{d^2M}{dx^2} = ky - q. \tag{c}$$

Using now the known differential equation of a beam in bending, $EI(d^2y/dx^2) = -M$, and differentiating it twice, we obtain

$$EI\frac{d^4y}{dx^4} = -\frac{d^2M}{dx^2}. \tag{d}$$

Hence by using (c) we find

$$EI\frac{d^4y}{dx^4} = -ky + q. \tag{1}$$

This is the differential equation for the deflection curve of a beam supported on an elastic foundation. Along the unloaded parts of the beam, where no distributed load is acting, $q = 0$, and the equation above will take the form

$$EI\frac{d^4y}{dx^4} = -ky. \tag{2}$$

It will be sufficient to consider below only the general solution of (2), from which solutions will be obtained also for cases implied in (1) by adding to it a particular integral corresponding to q in (1).

Substituting $y = e^{mx}$ in (2), we obtain the characteristic equation

$$m^4 = -\frac{k}{EI},$$

which has the roots

$$m_1 = -m_3 = \sqrt[4]{\frac{k}{4EI}}\,(1 + i) = \lambda(1 + i),$$

$$m_2 = -m_4 = \sqrt[4]{\frac{k}{4EI}}\,(-1 + i) = \lambda(-1 + i).$$

The general solution of (2) takes the form

$$y = A_1e^{m_1x} + A_2e^{m_2x} + A_3e^{m_3x} + A_4e^{m_4x}, \tag{e}$$

where

$$\lambda = \sqrt[4]{\frac{k}{4EI}}. \tag{f}$$

Using

$$e^{i\lambda x} = \cos \lambda x + i \sin \lambda x,$$

$$e^{-i\lambda x} = \cos \lambda x - i \sin \lambda x,$$

and introducing the new constants C_1, C_2, C_3, and C_4, where

$$(A_1 + A_4) = C_1, \qquad i(A_1 - A_4) = C_2,$$

$$(A_2 + A_3) = C_3, \qquad i(A_2 - A_3) = C_4,$$

we can write (e) in a more convenient form:

$$y = e^{\lambda x}(C_1 \cos \lambda x + C_2 \sin \lambda x) + e^{-\lambda x}(C_3 \cos \lambda x + C_4 \sin \lambda x). \tag{3a}$$

Here λ includes the flexural rigidity of the beam as well as the elasticity of the supporting medium, and is an important factor influencing the shape of the elastic line. For this reason the factor λ is called the *characteristic* of the system, and, since its dimension is length^{-1}, the term $1/\lambda$ is frequently referred to as the *characteristic length*. Consequently, λx will be an absolute number.

Expression (3a) represents the general solution for the deflection line of a straight prismatic bar supported on an elastic foundation and subjected to transverse bending forces, but with no q loading. An additional term is necessary where there is a distributed load. By differentiation of (3a) we get

$$\left.\begin{aligned}\frac{1}{\lambda}\frac{dy}{dx} &= e^{\lambda x}[C_1(\cos\lambda x - \sin\lambda x) + C_2(\cos\lambda x + \sin\lambda x)] \\ &\qquad - e^{-\lambda x}[C_3(\cos\lambda x + \sin\lambda x) - C_4(\cos\lambda x - \sin\lambda x)], \\ \frac{1}{2\lambda^2}\frac{d^2y}{dx^2} &= -e^{\lambda x}(C_1\sin\lambda x - C_2\cos\lambda x) + e^{-\lambda x}(C_3\sin\lambda x - C_4\cos\lambda x) \\ \frac{1}{2\lambda^3}\frac{d^3y}{dx^3} &= -e^{\lambda x}[C_1(\cos\lambda x + \sin\lambda x) - C_2(\cos\lambda x - \sin\lambda x)] \\ &\qquad + e^{-\lambda x}[C_3(\cos\lambda x - \sin\lambda x) + C_4(\cos\lambda x + \sin\lambda x)].\end{aligned}\right\} \quad (3\,\text{b–d})$$

Knowing that

$$\frac{dy}{dx} = \tan\theta, \quad -EI\frac{d^2y}{dx^2} = M, \quad \text{and} \quad -EI\frac{d^3y}{dx^3} = Q, \qquad (g)$$

we can obtain the general expressions for the slope θ* of the deflection line as well as for the bending moment M and the shearing force Q from (3 b–d). The intensity of pressure in the foundation will be found from (3a) to be $p = ky$.

In applying these general equations, or corresponding ones including the term dependent on q, to particular cases the next step is to determine the constants of integration C_1, C_2, C_3, and C_4. These integration constants depend on the manner in which the beam is subjected to the loading and have constant values along each portion of the beam within which the elastic line and all its derivatives are continuous. Their values can be obtained from the conditions existing at the two ends of such continuous portions. Out of the four quantities (y, θ, M, and Q) characterizing the condition of an end, two are usually known at each end, from which sufficient data are furnished for the determination of the constants C.

When a beam is subjected to various loads the elastic line must be resolved into continuous portions (for example, A–a, a–b, b–c, and c–B in Fig. 1); then at the intermediate points the consideration of the material continuity of the beam will furnish the data for determining the integration constants for each of these portions.

Although from the point of view of mathematics the problem can be completely solved in this way,† the procedure is laborious and not well fitted to practical computation. The work can be considerably simplified, however, if the general solution is written in such a form that the integration constants obtain a physical interpretation in terms of the end conditions. This method of solution will be discussed in the next section.

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* On the basis of the approximate bending formula used above in the derivation (d) it is permissible to put $\tan\theta = \theta$.

† This method was used by K. Hayashi in his book *Theorie des Trägers auf elastischer Unterlage und ihre Anwendung auf den Tiefbau* (Berlin, 1921).

2. *Interpretation of the Integration Constants*

Assume a beam subjected to various loading (such as moment M, force P, and distributed load q) and take the origin of an x, y coordinate system at the left end of the beam (Fig. 3).

In (3 a–d) general expressions were obtained for the y, θ, M, and Q quantities of a beam in bending. Taking in these equations $x = 0$, we get the conditions at the left end of our beam as

$$\left.\begin{aligned} [y]_{x=0} &= y_0 = C_1 + C_3\,, \\ \left[\frac{dy}{dx}\right]_{x=0} &= \theta_0 = \lambda(C_1 + C_2 - C_3 + C_4), \\ \left[-EI\frac{d^2y}{dx^2}\right]_{x=0} &= M_0 = 2\lambda^2 EI(-C_2 + C_4), \\ \left[-EI\frac{d^3y}{dx^3}\right]_{x=0} &= Q_0 = 2\lambda^3 EI(C_1 - C_2 - C_3 - C_4). \end{aligned}\right\} \tag{a}$$

Expressing the C's as unknowns, we have, from the equations above,

$$\left.\begin{aligned} C_1 &= \tfrac{1}{2}y_0 + \frac{1}{4\lambda}\theta_0 + \frac{1}{8\lambda^3 EI}Q_0\,, \\ C_2 &= \frac{1}{4\lambda}\theta_0 - \frac{1}{4\lambda^2 EI}M_0 - \frac{1}{8\lambda^3 EI}Q_0\,, \\ C_3 &= \tfrac{1}{2}y_0 - \frac{1}{4\lambda}\theta_0 - \frac{1}{8\lambda^3 EI}Q_0\,, \\ C_4 &= \frac{1}{4\lambda}\theta_0 + \frac{1}{4\lambda^2 EI}M_0 - \frac{1}{8\lambda^3 EI}Q_0\,. \end{aligned}\right\} \tag{b}$$

Substituting these expressions for the C's in (3a) and putting $\frac{1}{2}(e^{\lambda x} + e^{-\lambda x}) = \text{Cosh}\,\lambda x$ and $\frac{1}{2}(e^{\lambda x} - e^{-\lambda x}) = \text{Sinh}\,\lambda x$, we find that the general equation of the elastic line will take the form

$$y_x = y_0 F_1(\lambda x) + \frac{1}{\lambda}\theta_0 F_2(\lambda x) - \frac{1}{\lambda^2 EI}M_0 F_3(\lambda x) - \frac{1}{\lambda^3 EI}Q_0 F_4(\lambda x)\cdots, \tag{c}$$

where

$$\begin{aligned} F_1(\lambda x) &= \text{Cosh}\,\lambda x \cos\lambda x, \\ F_2(\lambda x) &= \tfrac{1}{2}(\text{Cosh}\,\lambda x \sin\lambda x + \text{Sinh}\,\lambda x \cos\lambda x), \\ F_3(\lambda x) &= \tfrac{1}{2}\text{Sinh}\,\lambda x \sin\lambda x, \\ F_4(\lambda x) &= \tfrac{1}{4}(\text{Cosh}\,\lambda x \sin\lambda x - \text{Sinh}\,\lambda x \cos\lambda x). \end{aligned}$$

It is seen that in (c) the general solution was put in a form in which the previous integration constants were replaced by the y_0, θ_0, M_0, and Q_0 quantities existing at the end $x = 0$ of the beam. On account of this feature the method developed on the basis of (c) is termed the *method of initial conditions*;* because the simple interpretation of the integration constants it has a considerable advantage over the method outlined in the previous section.

A more generalized form of (c) can be obtained through the following reasoning: Assume that the y_0, θ_0, M_0, and Q_0 quantities are known; then we can proceed from the left end of the beam toward the right along the unloaded portion A–a until we arrive at the point where the first load is applied to the beam. Assume that the first loading is a concentrated moment M, as shown in Figure 3. Evidently this moment M must have an effect to the right ($x > u_M$) of its point of application similar to that which the initial moment M_0 had on the A–a portion ($0 < x < u_M$) of the elastic line. Seeing from (c) that the factor of M_0 was $-(1/\lambda^2EI)F_3(\lambda x)$, we can conclude that the moment M at a should have a modifying effect of $-(1/\lambda^2EI)F_3[\lambda(x - u_M)]M$ on the elastic line to the right of point a, where $x > u_M$. Consequently, we obtain the deflection curve on the portion a–b by adding this last expression to (c).

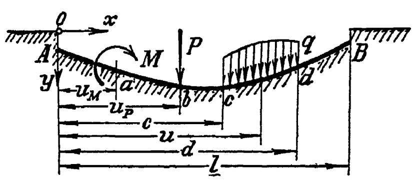

FIG. 3

In a similar way we find that the force P will have an influence $(1/\lambda^3EI)F_4[\lambda(x - u_P)]P$† on the deflection line to the right of point b. Finally, since the distributed loading q can be regarded as consisting of infinitesimal concentrated forces, we can conclude that its effect for the $x > c$ portion must be $(1/\lambda^3EI)\int_c^x qF_4[\lambda(x - u)]\,du$. For $x > d$ the upper limit of the integral becomes d. Summing up these results, we find the equation of the deflection line for such a case as that shown in Figure 3‡ to be

$$y_x = y_0F_1(\lambda x) + \frac{1}{\lambda}\theta_0F_2(\lambda x) - \frac{1}{\lambda^2EI}M_0F_3(\lambda x) - \frac{1}{\lambda^3EI}Q_0F_4(\lambda x)$$
$$- \frac{1}{\lambda^2EI}MF_3[\lambda(x - u_M)] + \frac{1}{\lambda^3EI}PF_4[\lambda(x - u_P)]$$
$$+ \frac{1}{\lambda^3EI}\int_c^x qF_4[\lambda(x - u)]\,du. \qquad (4a)$$

* This method was developed largely in Russia. See A. A. Umansky, *Analysis of Beams on Elastic Foundation*, Central Research Institute of Auto-Transportation (Leningrad, 1933); and *idem*, *Special Course in Structural Mechanics*, General Redaction of Literature of Building (Leningrad-Moscow, 1935), Part I. These publications contain also bibliographies of earlier Russian works.

† Here the sign of the term taken from (c) had to be changed, since the downward-acting force P represents a negative shear for the portion to the right of point b.

‡ The expression for the deflection line could be generalized in a still larger sense by including among the loadings concentrated changes in the deflection ordinates and in the slopes, and also by regarding distributed moments as a loading type. Expressions for such cases are to be found in Umansky, *opera cit.*

This equation includes the effect of M, P, and q acting on the beam between the left end ($x = 0$) and the point under consideration ($x = x$). If any of these loadings are absent on this portion of the beam the corresponding term in (4a) should be disregarded. By taking the consecutive derivatives of the equation above and putting

$$\frac{dF_1}{dx} = -4\lambda F_4, \qquad \frac{dF_2}{dx} = \lambda F_1, \qquad \frac{dF_3}{dx} = \lambda F_2, \quad \text{and} \quad \frac{dF_4}{dx} = \lambda F_3,$$

we obtain the expressions below for slope, moment, and shearing force:

$$\left.\begin{aligned}
\theta_x &= \theta_0 F_1(\lambda x) - \frac{1}{\lambda EI} M_0 F_2(\lambda x) - \frac{1}{\lambda^2 EI} Q_0 F_3(\lambda x) - 4\lambda y_0 F_4(\lambda x) \\
&\quad - \frac{1}{\lambda EI} M F_2[\lambda(x - u_M)] + \frac{1}{\lambda^2 EI} P F_3[\lambda(x - u_P)] \\
&\quad + \frac{1}{\lambda^2 EI} \int_c^x q F_3[\lambda(x - u)]\, du, \\
M_x &= M_0 F_1(\lambda x) + \frac{1}{\lambda} Q_0 F_2(\lambda x) + \frac{k}{\lambda^2} y_0 F_3(\lambda x) + \frac{k}{\lambda^3} \theta_0 F_4(\lambda x) \\
&\quad + M F_1[\lambda(x - u_M)] - \frac{1}{\lambda} P F_2[\lambda(x - u_P)] - \frac{1}{\lambda} \int_c^x q F_2[\lambda(x - u)]\, du, \\
Q_x &= Q_0 F_1(\lambda x) + \frac{k}{\lambda} y_0 F_2(\lambda x) + \frac{k}{\lambda^2} \theta_0 F_3(\lambda x) - 4\lambda M_0 F_4(\lambda x) \\
&\quad - 4\lambda M F_4[\lambda(x - u_M)] - P F_1[\lambda(x - u_P)] - \int_c^x q F_1[\lambda(x - u)]\, du.
\end{aligned}\right\} \quad (4\,\text{b–d})$$

It is seen that the initial conditions appear in equations (4 a–d) according to a systematic scheme. In each of these equations all the four initial conditions are present and the order of their succession is shifted by one place at a time as we proceed from (4a) to (4d). The same systematic shifting can be observed also in the F functions connected with the loading terms M, P, and q.

Putting $x = l$ into (4 a–d), we obtain the y_l, θ_l, M_l, and Q_l quantities for the right end of the beam as expressed in terms of the initial conditions and the loadings. These relations can then be used to determine the unknown initial conditions. As we have said, out of the four quantities which define the condition of one end of a beam, two are usually known at each end in every case. There remain two unknowns at each end; altogether there are four unknown quantities which can be determined from (4 a–d).

Consider for instance the beam in Figure 3 with both ends free. Here we have $M_0 = 0$, $Q_0 = 0$ and $M_l = 0$, $Q_l = 0$. Substituting these values in (4 a–d) we find that the left-hand side of (4c) and (4d) will be zero, while the right-hand side will contain only two unknown initial conditions, y_0 and θ_0. From the two simultaneous equations the two unknown quantities can be determined; then, substituting these, in turn, in the general expressions (4 a–d), we can proceed to calculate the y_x, θ_x, M_x, and Q_x values for any intermediate point on the beam. The outstanding feature of this method is the simple physical interpretation of the integration constants and the systematic order in which these constants appear in the equations. For practical computation, however, the method can be considered only if there are numerical tables of the F functions available, and even then more complicated loadings involve lengthy and intricate calculations.

3. *Method of Superposition*

In the preceding sections two different methods have been presented, both of them aiming to determine the integration constants from the prescribed end conditions of the elastic line. It has been seen that the main difficulty in applying the general solution to particular problems arises in the determination of the integration constants, which involves a considerable amount of work in both methods discussed.

These difficulties can be largely avoided by using the *method of superposition.** The advantage of this method lies in the fact that the determination of the integration constants for a beam of unlimited length (an infinitely long beam) is very simple and that, consequently, the equation of the deflection line for any loading on the infinitely long beam can be obtained in a concise form. Such deflection formulas will be derived in Chapter II; in Chapter III it will be shown that by superposing the formulas obtained for the infinitely long beam solutions can be derived for beams of any length and with any loading and end conditions. This procedure will prove to be the simplest in the application to particular problems; it can be used also when, in addition to the lateral loads, axial forces or twisting moments are acting on the beam.†

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* The application of the method of superposition in the solution of beams of finite length on an elastic foundation was first proposed by the writer in a paper called "Analysis of Bars on Elastic Foundation," *Final Report of the Second International Congress for Bridge and Structural Engineering* (Berlin-Munich, 1936).

† The scheme in the method of initial conditions loses its periodical character when axial forces, in addition to the transverse loading, are acting on the beam.

CHAPTER II

BEAMS OF UNLIMITED LENGTH

I. The Infinite Beam

4. *Concentrated Loading*

Consider a beam of unlimited length in both directions (an infinite beam) subjected to a single concentrated force P at point O (Fig. 4). Because of the apparent symmetry of the deflection curve we need to consider only the half which is to the right of point O, the origin of the x, y rectangular coordinate system.

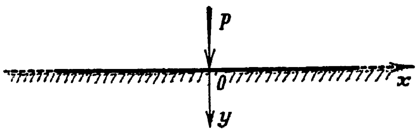

FIG. 4

In §1 we found that the general solution for the deflection curve of a beam subjected to transverse loading can be written as equation (3a):

$$y = e^{\lambda x}(C_1 \cos \lambda x + C_2 \sin \lambda x) + e^{-\lambda x}(C_3 \cos \lambda x + C_4 \sin \lambda x). \tag{a}$$

In the present problem, dealing with a beam of unlimited length, it is reasonable to assume that in an infinite distance from the application of the load the deflection of the beam must approach zero, that is, if $x \to \infty$, then $y \to 0$. This condition can be fulfilled only if in the equation above the terms connected with $e^{\lambda x}$ vanish, which necessitates that in the case under discussion $C_1 = 0$ and $C_2 = 0$. Hence the deflection curve for the right part ($x > 0$) of the beam will take the form

$$y = e^{-\lambda x}(C_3 \cos \lambda x + C_4 \sin \lambda x). \tag{b}$$

From the condition of symmetry we know that

$$\left[\frac{dy}{dx}\right]_{x=0} = 0,$$

that is, $-(C_3 - C_4) = 0$, from which we find $C_3 = C_4 = C$. This last constant of the equation

$$y = Ce^{-\lambda x} (\cos \lambda x + \sin \lambda x) \tag{c}$$

can be obtained from the consideration that the sum of the reaction forces will keep equilibrium with the load P, that is,

$$2\int_0^\infty ky\,dx = P.$$

Since $2kC \int_0^\infty e^{-\lambda x}(\cos \lambda x + \sin \lambda x)\, dx = 2kC(1/\lambda)$, from $2kC(1/\lambda) = P$ we get $C = P\lambda/2k$, and, substituting this in (c) above, we have

$$y = \frac{P\lambda}{2k} e^{-\lambda x} (\cos \lambda x + \sin \lambda x), \tag{d}$$

which gives the deflection curve for the right side ($x \geqq 0$) of the beam. This deflection curve is a wavy line with decreasing amplitude (Fig. 5a). The deflection under the load is $y_0 = P\lambda/2k$; the zero points of the line are where $\cos \lambda x + \sin \lambda x = 0$, that is, at the consecutive values of $\lambda x = \frac{3}{4}\pi, \frac{7}{4}\pi, \frac{11}{4}\pi$, etc.

Taking the successive derivatives of y (see [d]) with respect to x, we obtain the expressions for θ, M, and Q on the right side of the beam as

$$\left.\begin{aligned} \frac{dy}{dx} &= \theta = -\frac{P\lambda^2}{k} e^{-\lambda x} \sin \lambda x, \\ -EI\frac{d^2y}{dx^2} &= M = \frac{P}{4\lambda} e^{-\lambda x}(\cos \lambda x - \sin \lambda x), \\ -EI\frac{d^3y}{dx^3} &= Q = -\frac{P}{2} e^{-\lambda x} \cos \lambda x. \end{aligned}\right\} \tag{e–g}$$

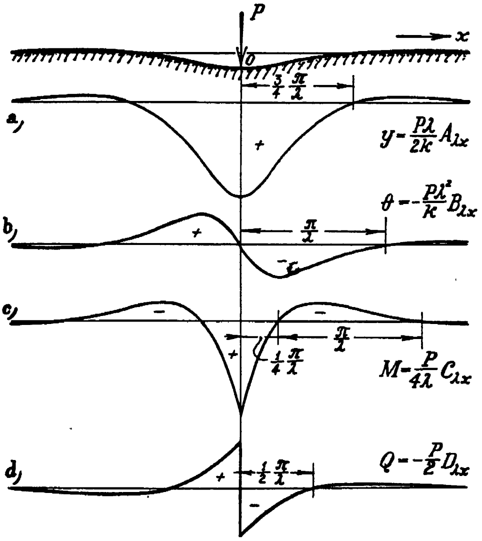

FIG. 5

The curves represented by the equations above are shown in Figure 5. They have all the features of damped waves.* At the point of application of the load ($x = 0$) or, to be precise, infinitely close to the right of it, we have the values $\theta = 0$, $M = P/4\lambda$, and $Q = -P/2$. In the derivation of the general solution for the elastic line (see p. 3) the positive directions were defined for the shearing force Q (positive when acting upward on the left of the elemental section) and for the bending moment M (the moment on the left of the element in the direction of the positive shearing force). As an extension of this convention, we shall regard as positive quantities the downward-acting loading (P), downward deflection (y), and the angular deflection (θ) rotating clockwise. Equations (d–g) give the

* This is the reason why the characteristic λ is sometimes called the *damping factor*.

values of y, θ, M, and Q for the right side ($x > 0$) of the beam according to this practice. On the left side the y and M curves will keep the same signs ($y_x = y_{-x}$ and $M_x = M_{-x}$), but θ and Q will change their signs ($\theta_x = -\theta_{-x}$ and $Q_x = -Q_{-x}$), as is shown in Figure 5. In calculating the ordinates of these curves the value x is always taken as positive.

Introducing into (d–g), the symbols

$$\left.\begin{aligned} e^{-\lambda x}(\cos\lambda x + \sin\lambda x) &= A_{\lambda x}, & e^{-\lambda x}\sin\lambda x &= B_{\lambda x}, \\ e^{-\lambda x}(\cos\lambda x - \sin\lambda x) &= C_{\lambda x}, & e^{-\lambda x}\cos\lambda x &= D_{\lambda x}, \end{aligned}\right\} \tag{h}$$

we can write

$$\left.\begin{aligned} y &= \frac{P\lambda}{2k} A_{\lambda x}, \\ \theta &= -\frac{P\lambda^2}{k} B_{\lambda x}, \\ M &= \frac{P}{4\lambda} C_{\lambda x}, \\ Q &= -\frac{P}{2} D_{\lambda x}. \end{aligned}\right\} \tag{5 a–d}$$

The A, B, C, and D functions of λx above will be frequently used in the rest of the text when solving different problems connected with the theory of beams on elastic foundation; therefore, in order to facilitate their application, tables are attached at the end of this work, giving the A_x, B_x, C_x, and D_x quantities as functions of a parameter x.* These functions are essentially all of the same type, since by properly shifting the origin we can obtain one from the other. Through simple trigonometrical transformation we can find that

$$A_{\lambda x} = -2\gamma B_{\lambda x_3} = \beta C_{\lambda x_2} = 2\alpha D_{\lambda x_1}, \tag{i}$$

where

$$\lambda x_1 = \lambda x - \pi/4, \qquad \lambda x_2 = \lambda x - \pi/2, \qquad \lambda x_3 = \lambda x - 3\pi/4,$$

and

$$\alpha = e^{-\pi/4}\sin\pi/4, \qquad \beta = e^{-\pi/2}\sin\pi/2, \qquad \gamma = e^{-3\pi/4}\sin 3\pi/4.$$

These four functions form a periodical scheme We find that

$$\left.\begin{aligned} \frac{d}{dx}A_{\lambda x} &= -2\lambda B_{\lambda x}, & \frac{d}{dx}B_{\lambda x} &= \lambda C_{\lambda x}, \\ \frac{d}{dx}C_{\lambda x} &= -2\lambda D_{\lambda x}, \text{ and } & \frac{d}{dx}D_{\lambda x} &= -\lambda A_{\lambda x}. \end{aligned}\right\} \tag{j}$$

* Such numerical tables were first given by H. Zimmermann in his principal book on this subject, *Die Berechnung des Eisenbahnoberbaues* (Berlin, 1888; 2d ed., Berlin, 1930).

The zero points for A are located at $\lambda x = (\frac{3}{4} + n)\pi$, for B at $\lambda x = n\pi$, for C at $\lambda x = (\frac{1}{4} + n)\pi$, and for D at $\lambda x = (\frac{1}{2} + n)\pi$; similarly, the extreme values (maximum or minimum) of these functions are found (outside of the origin) for A at $\lambda x = n\pi$, for B at $\lambda x = (\frac{1}{4} + n)\pi$, for C at $\lambda x = (\frac{1}{2} + n)\pi$, and for D at $\lambda x = (\frac{3}{4} + n)\pi$, where $n = 0, 1, 2, 3$, etc

An important feature of these functions is the rapidly decreasing amplitude. As may be seen from the appended tables (see pp. 217–239), when $\lambda x > 1.5\pi$ the value of any of the four functions is under 0.01. This means that the manner in which the beam is supported in a distance of $x = 1.5\pi/\lambda$ (or even $x = \pi/\lambda$) from the application of the load will have only a small effect on the formation of the deflection line, or, in other words, a beam of the length $l = 2\pi/\lambda$ loaded with a concentrated force P at the middle will exhibit approximately the same deflection curve as the infinitely long beam shown in Figure 5. Since elastically supported beams with large λl values are frequently applied in engineering constructions, these considerations will permit us to derive approximate solutions for such problems, using the results obtained in this chapter for beams of unlimited length.

Returning to the discussion of the curves given by (5 a–d), we find that y as well as θ, M, and Q is *proportional* to the loading P; hence it follows that the *principle of superposition* and the *reciprocity theorem* are directly applicable to the system. We encounter here the most general applicability of the reciprocal theorem, since it holds in this case not only for displacements, but also for angular deflections, bending moments, and shearing forces. If at point 1 a force P_1 and at point 2 a force P_2 are acting, it is apparent either from the curves of Figure 5 or from (5 a–d) that $y_{1,2} = y_{2,1}$, $\theta_{1,2} = \pm\theta_{2,1}$;* furthermore, $M_{1,2} = M_{2,1}$ and $Q_{1,2} = \pm Q_{2,1}$.* This states the reciprocity theorem in the most general form and proves that the y, θ, M, and Q curves of Figure 5 are at the same time *influence lines* for deflection, angular deflection, moment, and shear.

So far we have been dealing with the situation in which a single concentrated force P is acting on the beam. From the formulas here obtained the deflection line can be derived also for the case when a concentrated moment M_0 is applied at point O on the infinitely long beam (Fig. 6a). This concentrated moment can be regarded as a limiting case of the loading shown in Figure 6b, if we assume that while a is approaching zero ($a \to 0$) Pa will simultaneously approach the value of M_0 ($Pa \to M_0$).

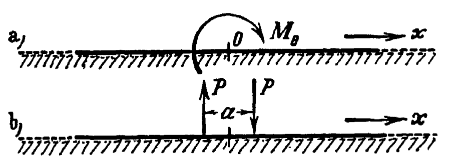

FIG. 6

Using (5a), we can write the deflection line for the loading in Figure 6b as

$$y = \frac{P\lambda}{2k}(-A_{\lambda(x+a)} + A_{\lambda x}) = -\frac{Pa\lambda}{2k}\frac{A_{\lambda(x+a)} - A_{\lambda x}}{a} \quad \text{for} \quad x > 0.$$

* Plus–minus sign according to the convention adopted (see p. 11).

Since

$$\left[\frac{A_{\lambda(x+a)} - A_{\lambda x}}{a}\right]_{a\to 0} = \frac{d}{dx} A_{\lambda x} = -2\lambda B_{\lambda x},$$

and at the same time

$$[Pa]_{a\to 0} = M_0,$$

we get the deflection line due to the M_0 clockwise moment as

$$y = \frac{M_0\lambda^2}{k} B_{\lambda x}. \tag{6a}$$

Taking the consecutive derivatives of y, we have

$$\left.\begin{aligned} \frac{dy}{dx} &= \theta = \frac{M_0\lambda^3}{k} C_{\lambda x}, \\ -EI\frac{d^2y}{dx^2} &= M = \frac{M_0}{2} D_{\lambda x}, \\ -EI\frac{d^3y}{dx^3} &= Q = -\frac{M_0\lambda}{2} A_{\lambda x}. \end{aligned}\right\} \tag{6 b-d}$$

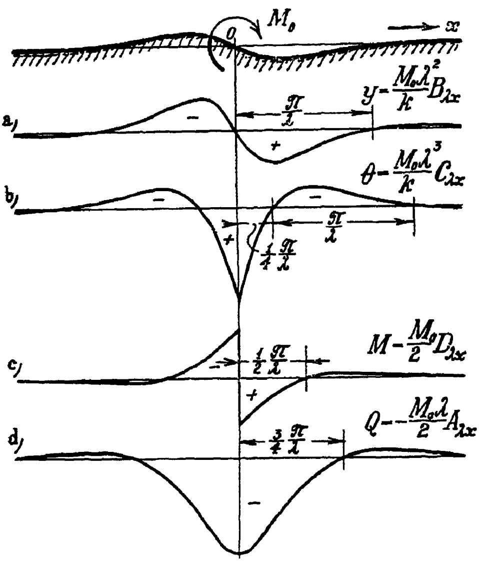

FIG. 7

The equations above present the y, θ, M, and Q curves, according to the sign convention established, for the right side ($x > 0$) of the beam. To the left of the point of application of the moment M_0 the sign of y and M must be reversed (see Fig. 7). When the M_0 moment is acting counterclockwise, opposite to the direction taken here, naturally the signs of all the curves in Figure 7 on both sides must be reversed. The arguments of the functions are always taken as positive, no matter on which side the point under consideration is located.

5. *Uniformly Distributed Loading*

Consider a uniform loading distributed over an A–B portion of the infinitely long beam and find the effect of this loading at an arbitrary point C, which is a distance a from the left end of the loaded portion and a distance b from the right end.

The distributed loading can be regarded as consisting of infinitely small concentrated forces qdx. The deflection produced at point C by such an element can be obtained by substituting qdx for P in (d) on page 11, which gives

$$\delta y = \frac{qdx\lambda}{2k} e^{-\lambda x}(\cos \lambda x + \sin \lambda x), \tag{a}$$

where x denotes the distance of the qdx element from point C. The total effect of the distributed loading can be obtained by integrating the expressions above within the a–b limits of the loaded part of the beam. Here we shall distinguish three cases, based on whether the point C is (a) within the loaded part, (b) to the left of it, or (c) to the right of it.

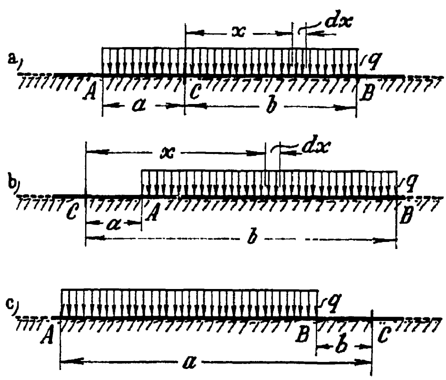

FIG. 8

a. When Point C is under the Loading (Fig. 8a)

In Figure 8a the distance a to the left can be regarded as negative, and so the deflection at C, as the integral of (a) above, will be

$$y_c = \frac{q\lambda}{2k}\left[\int_0^a e^{-\lambda x}(\cos \lambda x + \sin \lambda x)\,dx + \int_0^b e^{-\lambda x}(\cos \lambda x + \sin \lambda x)\,dx\right]$$

$$= \frac{q}{2k}\left[(1 - e^{-\lambda a}\cos \lambda a) + (1 - e^{-\lambda b}\cos \lambda b)\right]. \tag{b}$$

Hence we can write

$$y_c = \frac{q}{2k}(2 - D_{\lambda a} - D_{\lambda b}). \tag{7a}$$

In a similar way, substituting qdx for P in (5 b–d) and integrating within the assigned limits, we get expressions of θ, M, and Q at point C as

$$\left.\begin{aligned} \theta_c &= \frac{q\lambda}{2k}(A_{\lambda a} - A_{\lambda b}), \\ M_c &= \frac{q}{4\lambda^2}(B_{\lambda a} + B_{\lambda b}), \\ Q_c &= \frac{q}{4\lambda}(C_{\lambda a} - C_{\lambda b}). \end{aligned}\right\} \tag{7 b–d}$$

b. When Point C is to the Left of the Loading (Fig. 8b)

In the problem in Figure 8b both a and b distances will be considered positive. By the sign convention adopted on page 11 we obtain now by integration

$$\left.\begin{aligned} y_c &= \frac{q}{2k}(D_{\lambda a} - D_{\lambda b}), \\ \theta_c &= \frac{q\lambda}{2k}(A_{\lambda a} - A_{\lambda b}), \\ M_c &= -\frac{q}{4\lambda^2}(B_{\lambda a} - B_{\lambda b}), \\ Q_c &= \frac{q}{4\lambda}(C_{\lambda a} - C_{\lambda b}). \end{aligned}\right\} \quad (8\text{ a–d})$$

c. When Point C is to the Right of the Loading (Fig. 8c)

Formulas for the situation in Figure 8c can be obtained from (8 a–d) by interchanging there the symbols a and b, and, furthermore, reversing the signs of θ and Q, as was done in dealing with a concentrated loading.

In this way we get for the point C*

$$\left.\begin{aligned} y_c &= -\frac{q}{2k}(D_{\lambda a} - D_{\lambda b}), \\ \theta_c &= \frac{q\lambda}{2k}(A_{\lambda a} - A_{\lambda b}), \\ M_c &= \frac{q}{4\lambda^2}(B_{\lambda a} - B_{\lambda b}), \\ Q_c &= \frac{q}{4\lambda}(C_{\lambda a} - C_{\lambda b}). \end{aligned}\right\} \quad (9\text{ a–d})$$

The correctness of the formulas derived for the three different positions of point C can be checked by substituting $a = 0$ and $b = l$ in (7 a–d) and (8 a–d), and thus we get terms for point A from either side equally:

$$\left.\begin{aligned} y_A &= \frac{q}{2k}(1 - D_{\lambda l}), \qquad \theta_A = \frac{q\lambda}{2k}(1 - A_{\lambda l}), \qquad M_A = \frac{q}{4\lambda^2} B_{\lambda l}, \\ Q_A &= \frac{q}{4\lambda}(1 - C_{\lambda l}). \end{aligned}\right\} \quad (c)$$

* These formulas differ only in signs from the ones given by (8 a–d) and are reproduced here merely for convenience in deciding signs.

From (7 a–d) and (9 a–d), by substituting $b = 0$ and $a = l$, we obtain, in both ways again, the same expressions for point B:

$$\left.\begin{aligned} y_B &= \frac{q}{2k}(1 - D_{\lambda l}), \qquad \theta_B = -\frac{q\lambda}{2k}(1 - A_{\lambda l}), \qquad M_B = \frac{q}{4\lambda^2}B_{\lambda l}, \\ Q_B &= -\frac{q}{4\lambda}(1 - C_{\lambda l}). \end{aligned}\right\} \quad \text{(d)}$$

6. *Triangular Loading*

We can distinguish again three cases according to the position of the point C, where the effect of the loading is sought.

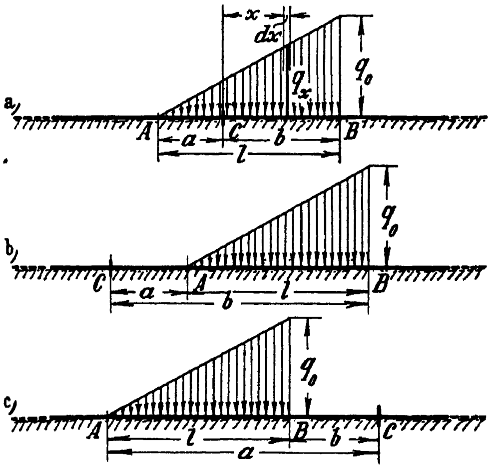

FIG. 9

a. *When Point C is under the Loading (Fig. 9a)*

Counting x from point C, we have in the region A–C in Figure 9a $q_x = (q_0/l)\ (a - x)$, and in the region C–B, $q_x = (q_0/l)\ (a + x)$, so that the total deflection at C will be

$$y_C = \frac{q_0\lambda}{2kl}\left\{\int_0^a (a - x)\, e^{-\lambda x}(\cos \lambda x + \sin \lambda x)dx + \int_0^b (a + x)\, e^{-\lambda x}(\cos \lambda x + \sin \lambda x)dx\right\}. \quad \text{(a)}$$

Carrying out this integration, we have

$$y_C = \frac{q_0}{4\lambda k}\frac{1}{l}(C_{\lambda a} - C_{\lambda b} - 2\lambda l D_{\lambda b} + 4\lambda a), \quad \text{(10a)}$$

and in a similar way we obtain

$$\left.\begin{aligned} \theta_C &= -\frac{q_0}{2k}\frac{1}{l}(D_{\lambda a} + D_{\lambda b} + \lambda l A_{\lambda b} - 2), \\ M_C &= -\frac{q_0}{8\lambda^3}\frac{1}{l}(A_{\lambda a} - A_{\lambda b} - 2\lambda l B_{\lambda b}), \\ Q_C &= \frac{q_0}{4\lambda^2}\frac{1}{l}(B_{\lambda a} + B_{\lambda b} - \lambda l C_{\lambda b}). \end{aligned}\right\} \quad \text{(10 b–d)}$$

In the same way, by integrating between the limits a and b, with attention to signs, we can obtain expressions for the other positions of point C. Below, only the results of these integrations are given for the second and third positions.

b. When Point C is to the Left of the Loading (Fig. 9b)

$$\left.\begin{aligned} y_C &= \frac{q_0}{4\lambda k}\frac{1}{l}(C_{\lambda a} - C_{\lambda b} - 2\lambda l D_{\lambda b}), \\ \theta_C &= \frac{q_0}{2k}\frac{1}{l}(D_{\lambda a} - D_{\lambda b} - \lambda l A_{\lambda b}), \\ M_C &= -\frac{q_0}{8\lambda^3}\frac{1}{l}(A_{\lambda a} - A_{\lambda b} - 2\lambda l B_{\lambda b}), \\ Q_C &= -\frac{q_0}{4\lambda^2}\frac{1}{l}(B_{\lambda a} - B_{\lambda b} + \lambda l C_{\lambda b}). \end{aligned}\right\} \tag{11 a–d}$$

c. When Point C is to the Right of the Loading (Fig. 9c)

$$\left.\begin{aligned} y_C &= \frac{q_0}{4\lambda k}\frac{1}{l}(C_{\lambda a} - C_{\lambda b} + 2\lambda l D_{\lambda b}), \\ \theta_C &= -\frac{q_0}{2k}\frac{1}{l}(D_{\lambda a} - D_{\lambda b} + \lambda l A_{\lambda b}), \\ M_C &= -\frac{q_0}{8\lambda^3}\frac{1}{l}(A_{\lambda a} - A_{\lambda b} + 2\lambda l B_{\lambda b}), \\ Q_C &= \frac{q_0}{4\lambda^2}\frac{1}{l}(B_{\lambda a} - B_{\lambda b} - \lambda l C_{\lambda b}). \end{aligned}\right\} \tag{12 a–d}$$

A trapezoidal loading can be obtained by superposing two reversed triangles of different heights. If the two triangles are of the same height we get a uniformly distributed loading over the portion A–B.

7. *Various Loading Conditions on the Infinite Beam*

When the infinitely long beam is subjected to a group of concentrated forces, then, according to the previously established law of superposition, the deflection, slope, M and Q, at any point can be obtained by summing up the effect of the separated forces on the point under consideration.

The deflection, for instance, for any point can be obtained by use of (5a), as

$$y = \frac{\lambda}{2k}\sum_{n=1}^{n} P_n A_{\lambda x_n}, \tag{a}$$

where x_n means the absolute distance of the force P_n from the cross section where the deflection is sought.

Let us assume that positive and negative forces are acting on a beam (Fig. 10) and denote the upward-acting negative forces by R. The deflection for any point can now be written as

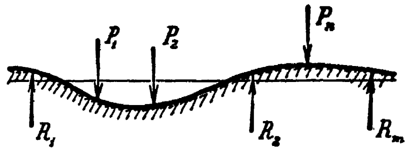

FIG. 10

$$y = \frac{\lambda}{2k}\left(\sum_{n=1}^{n} P_n A_{\lambda x_n} - \sum_{m=1}^{m} R_m A_{\lambda x_m}\right). \tag{13}$$

The R forces can be looked upon as concentrated reactions, and their values can be determined from the conditions assigned to the points where they are applied. We may require, for instance, that at any i point where such an R_i force is acting the deflection should be zero ($y_i = 0$), or we may assign such a condition that at each of these points the reaction R_i should be proportional to the deflection y_i occurring at that point, that is, $y_i = R_i/\Delta_i$, where Δ_i (spring constant) denotes the elasticity of the ith support.

In each of the cases above we can write as many equations of the type of (13) as there are R forces acting on the beam, and from that system of simultaneous equations the values of the unknown R's can be determined.

Such a system of equations can be solved most easily by a method of successive approximation.* Since the deflection line due to a single concentrated force has a rapidly decreasing amplitude (Fig. 5a), at the ith point the R_i force will have the greatest influence on the y_i deflection, and hence it is permissible, for the first approximation, to neglect the effect of the other R unknowns.

Making this first approximation, we shall find the ith equation to be

$$y_i = \frac{\lambda}{2k}\left(\sum_{n=1}^{n} P_n A_{\lambda x_n} - R_i\right), \tag{b}$$

where $y_i = 0$ (rigid support) or $y_i = R_i/\Delta_i$ (elastic support). In both cases the equation above for y_i will contain only one unknown, that is, R_i, and in this way the first approximate values of all R's can be obtained from a corresponding number of independent equations. By means of these first approximate values the process outlined above can be repeated until the required accuracy is obtained.

In the same way solutions can be derived when the bar is subjected to a combination of concentrated and distributed loads and when certain conditions are assigned to the elastic line simultaneously at a number of points. This method, based on the principle of superposition and on the damped-wave character of the elastic line, is adaptable to obtaining solutions for all problems of the type.

* See E. T. Whittaker and G. Robinson, *The Calculus of Observations* (London, 1924), p. 255.

So far we have been discussing the effect of single concentrated forces and moments. From Figures 5 and 7 it may be seen that a concentrated force causes discontinuity in shear and that a concentrated moment causes discontinuity in moment at the point of its application. In order to complete the investigation in this respect let us examine now the type of loading which will cause discontinuity in slope, or in deflection, at the section where it is applied.* Here we must have recourse to the curves shown in Figure 5. That these curves represent influence lines for deflection, slope, moment, and shear has already been established, and use of this property of the y and θ curves was made in § 4. Now we shall utilize in the same way the M and Q curves.

As we know from the theory of structures, if at one point on a beam, between two neighboring cross sections, a concentrated angular distortion is produced, the deflection line originated thereby will give the influence line of bending moment for the point where this relative distortion was produced.† Such distortion can be caused at point O by two moments V, applied in the manner shown in Figure 11, where the beam is assumed to be hinged at O; so we can conclude that this type of loading, which we shall call *double moment*, will produce a deflection line similar to the M curve in Figure 5. Since

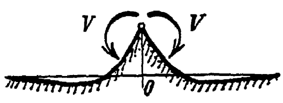

FIG. 11

$$y = Ce^{-\lambda x}(\cos \lambda x - \sin \lambda x), \tag{c}$$

the C constant can be determined from the condition of statics that

$$V = Ck \int_0^\infty xe^{-\lambda x} (\cos \lambda x - \sin \lambda x)dx = -Ck \frac{1}{2\lambda^2}. \tag{d}$$

Hence $C = -2V\lambda^2/k$, and we get the deflection line for the loading shown in Figure 11 as

$$y = -\frac{2V\lambda^2}{k} e^{-\lambda x} (\cos \lambda x - \sin \lambda x) = -\frac{2V\lambda^2}{k} C_{\lambda x}. \tag{14a}$$

From the consecutive derivatives we obtain for the right side ($x > 0$) of the beam

$$\left.\begin{aligned} \theta &= \frac{4V\lambda^3}{k} D_{\lambda x}, \\ M &= VA_{\lambda x}, \\ Q &= -2V\lambda B_{\lambda x}. \end{aligned}\right\} \tag{14 b-d}$$

* See the papers by P. Neményi: "Eine neue Singularitätenmethode für die Elastizitätstheorie," *Zeitschrift für angewandte Mathematik und Mechanik*, 9 (1929), 488–490; and "Tragwerke auf elastisch nachgiebiger Unterlage," in the same periodical, 11 (1931), 450–463.

† This statement can be proved by the reciprocity theorem. The theory, developed on similar considerations, is termed the *kinetic* (*kinematic*) *theory*. See the book by D. A. Molitor, *Kinetic Theory of Engineering Structures* (New York, 1911).

These curves are shown in Figure 12. We find that as $x \to 0$, $M \to V$ and $Q \to 0$.

In a similar way we can conclude that the influence line for shearing force is a deflection curve caused by a double shearing force W (Fig. 13) acting at point O, where the beam is assumed to be cut, and producing there between the two neighboring cross sections a distortion which is related to the shearing force. The deflection curve due to these W's will have the same form as the Q curve in Figure 5, that is,

$$y = Ce^{-\lambda x} \cos \lambda x. \qquad \text{(e)}$$

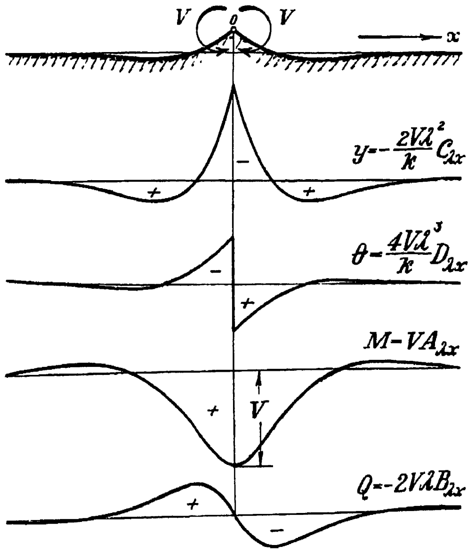

FIG. 12

From the consideration of equilibrium we know that

$$W = Ck \int_0^\infty e^{-\lambda x} \cos \lambda x dx = Ck \frac{1}{2\lambda}, \qquad \text{(f)}$$

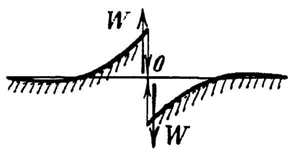

FIG. 13

from which $C = 2W\lambda/k$, and so we have for the right side of the beam

$$y = \frac{2W\lambda}{k} e^{-\lambda x} \cos \lambda x = \frac{2W\lambda}{k} D_{\lambda x}, \qquad \text{(15a)}$$

and from this,

$$\left.\begin{aligned} \theta &= -\frac{2W\lambda^2}{k} A_{\lambda x}, \\ M &= -\frac{W}{\lambda} B_{\lambda x}, \\ Q &= -WC_{\lambda x}. \end{aligned}\right\} \qquad \text{(15 b-d)}$$

The curves representing the equations above are shown in Figure 14.

The loadings V and W are the ones which produce concentrated change in slope and deflection, respectively. Their action has the following significance: if M is the bending moment caused by any loading at one section on the infinite beam and we apply to that point a double moment $V = -M$, as discussed above, we cancel the M moment at this section (producing, however, no change in shear), and the entire effect of these V forces will be the same as if we had inserted a hinge in the bar at the point where we made them act. We may therefore call such a place a *hinge for moment*, since it releases moments but

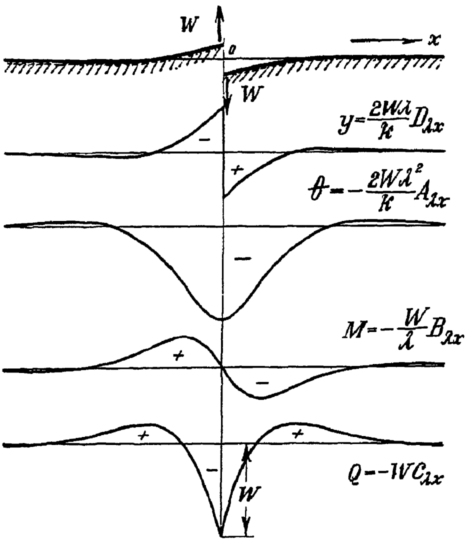

FIG. 14

resists shear. On the other hand, when we apply at one point the $W = -Q$ forces of the type discussed above, we cancel the shearing force which, because of some loading, existed at that point before, but produce no change in the bending moment. Hence we can say, by analogy, that now a *hinge for shear* has been introduced at the point.

We have now discussed all four types of loadings by which we can fulfill conditions required for deflection, slope, moment, or shear at any cross section of the infinite beam. It has been seen that these P, M, V, and W loadings correspond to discontinuities of the beam in shear, moment, slope, and deflection, respectively. By combining these loadings we can get a large variety of conditions.

II. The Semi-infinite Beam

8. *The End-Conditioning Forces*

The term *semi-infinite beam* will be used for a beam which has unlimited extension in only one direction, having at point A a finite end. This end may, under different conditions, be (*a*) free, (*b*) hinged, or (*c*) fixed (Fig. 15 b–d). It will be shown that under any loading or end conditions solution can be arrived at for the semi-infinite beam by using the formulas obtained previously for the infinite beam.

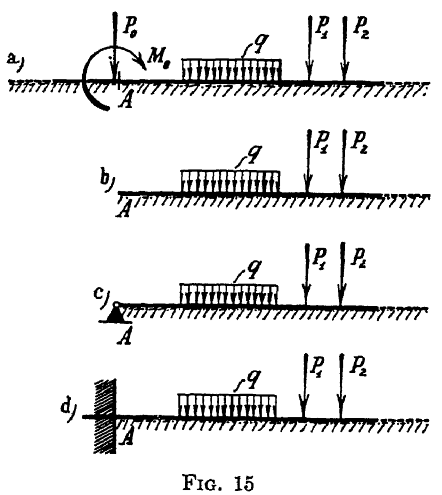

FIG. 15

a. Semi-infinite Beams with Free End (Fig. 15b)

Consider an infinitely long beam and assume that because of some loading we have at point A a bending moment M_A and a shearing force Q_A. Thus M_A and Q_A maintain the continuity of the beam at A. The removal of M_A and Q_A will have the same significance for the right side of the beam as the removal of the whole portion to the left. In other words, it will create a semi-infinite beam with a free end at point A. Hence our problem is reduced to making M_A and Q_A vanish on the infinite beam. This can be done by applying at A such a moment M_0 and force P_0 as will cause a bending moment $-M_A$ and a shearing force $-Q_A$ at that section. Because of the singularity in moment and shear (Figs. 5d and 7c) let us make the distinction that we require P_0 and M_0 to produce a moment and shear $-M_A$ and $-Q_A$ at the section infinitely close to the right of their point of application; or, in other words, we shall apply P_0 and M_0 infinitely close *to the left of point A* (Fig. 15a), where we want to cancel the moment and shear. According to the condition for a free end ($M = 0$ and $Q = 0$) we must have at A, using (5 c–d) and (6 c–d),

$$M_A + \frac{P_0}{4\lambda} + \frac{M_0}{2} = 0 \quad \text{and} \quad Q_A - \frac{P_0}{2} - \frac{M_0\lambda}{2} = 0, \tag{a}$$

getting thus

$$\left.\begin{aligned} P_0 &= 4(\lambda M_A + Q_A), \\ M_0 &= -\frac{2}{\lambda}(2\lambda M_A + Q_A). \end{aligned}\right\} \tag{16}$$

P_0 and M_0 applied on the infinite beam in the manner shown in Figure 15a will create at A the conditions of a free end ($M = 0$, $Q = 0$). Hence, under the action of the load, together with P_0 and M_0, the part of the infinitely long beam which is to the right of point A will behave in every respect as if there were a free end at A. Briefly, the situations shown in Figure 15 a–b are identical, provided P_0 and M_0 are determined from (16).

In a similar manner we can determine P_0 and M_0 so as to fulfill any other requirements at point A. In every case these forces must be applied at the place where we want the infinitely long bar to terminate. Since these quantities P_0 and M_0 *create* the required end conditions at the assigned point they will be referred to henceforth as the *end-conditioning forces.**

b. Semi-infinite Beam with Hinged End (Fig. 15c)

The conditions for A are $y = 0$ and $M = 0$. Hence if the load produces at this point a moment M_A and a deflection y_A, the end-conditioning forces P_0 and M_0 have to produce here $-M_A$ and $-y_A$ respectively. Using (5 a, c) and (6 a, c), we can write these conditions as

$$y_A + \frac{P_0\lambda}{2k} = 0 \quad \text{and} \quad M_A + \frac{P_0}{4\lambda} + \frac{M_0}{2} = 0, \tag{b}$$

from which we get the expressions for the end-conditioning forces:

$$\left.\begin{aligned} P_0 &= -\frac{2k}{\lambda} y_A, \\ M_0 &= \frac{k}{\lambda^2} y_A - 2M_A. \end{aligned}\right\} \tag{17}$$

* This will be used as a collective term, in which we shall include the moment M_0 as well as the force P_0.

Giving such values to P_0 and M_0, we shall obtain for the right side of the beam (Fig. 15a) a situation identical to that shown in Figure 15c.

c. Semi-infinite Beam with Fixed End (Fig. 15d)

In order to fulfill the conditions $y = 0$ and $\theta = 0$ for point A, if the loading produced y_A and θ_A at this point of the infinite beam the end-conditioning forces P_0 and M_0 would have to produce $-y_A$ and $-\theta_A$ at the same place. This condition can be written, by use of (5 a–b) and (6 a–b), as

$$y_A + \frac{P_0\lambda}{2k} = 0 \quad \text{and} \quad \theta_A + \frac{M_0\lambda^3}{k} = 0, \tag{c}$$

from which we obtain the end-conditioning forces as

$$\left.\begin{aligned} P_0 &= -\frac{2k}{\lambda}\, y_A\,, \\ M_0 &= -\frac{k}{\lambda^3}\,\theta_A\,. \end{aligned}\right\} \tag{18}$$

If P_0 and M_0 in Figure 15a have these values, the right part of the beam will behave in exactly the same manner as the fixed-end beam shown in Figure 15d.

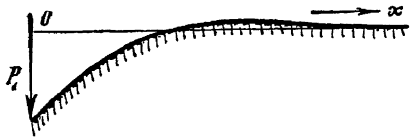

FIG. 16

9. *Particular Cases of End Loading*

The formulas developed above for the end-conditioning forces can also be used for deriving expressions for a semi-infinite beam subjected to various types of end conditions.

Let us consider first the situation shown in Figure 16. Here the end conditions for point O are $M = 0$ and $Q = -P_1$. Putting $M_A = 0$ and $Q_A = P_1$ into (16), we obtain the corresponding end-conditioning forces $P_0 = 4P_1$ and $M_0 = -(2/\lambda)P_1$. If we apply these P_0 and M_0 on the infinite beam, using (5 a–d) and (6 a–d), we get the solution for $x > 0$. The result will be

$$\left.\begin{aligned} y &= \frac{2P_1\lambda}{k} D_{\lambda x}\,, \\ \theta &= -\frac{2P_1\lambda^2}{k} A_{\lambda x}\,, \\ M &= -\frac{P_1}{\lambda} B_{\lambda x}\,, \\ Q &= -P_1 C_{\lambda x}\,. \end{aligned}\right\} \tag{19a–d}$$

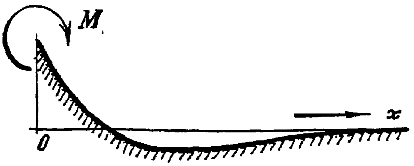

FIG. 17

In order to reach the solution for the situation shown in Figure 17 we have to put $M_A = -M_1$ and $Q_A = 0$ into (16), which then gives $P_0 = -4\lambda M_1$ and $M_0 = 4M_1$. Applying these end-conditioning forces at point O on the infinite beam, we have for values of $x > 0$:

$$\left.\begin{aligned} y &= -\frac{2M_1\lambda^2}{k} C_{\lambda x} , \\ \theta &= \frac{4M_1\lambda^3}{k} D_{\lambda x} , \\ M &= M_1 A_{\lambda x} , \\ Q &= -2M_1\lambda B_{\lambda x} . \end{aligned}\right\} \qquad (20\,\text{a-d})$$

If in the example above a hinged support is assumed at point O, then the corresponding end-conditioning forces are obtained by putting $y_A = 0$ and $M_A = -M_1$ into (17), which then yields $P_0 = 0$ and $M_0 = 2M_1$. These end-conditioning forces give the solution for the case shown in Figure 18:

FIG. 18

$$\left.\begin{aligned} y &= \frac{2M_1\lambda^2}{k} B_{\lambda x} , \\ \theta &= \frac{2M_1\lambda^3}{k} C_{\lambda x} , \\ M &= M_1 D_{\lambda x} , \\ Q &= -M_1\lambda A_{\lambda x} . \end{aligned}\right\} \qquad (21\text{ a–d})$$

In case of a displacement y_0 and a rotation θ_0 of the end $x = 0$ of a semi-infinite beam, the deflection and stresses can be derived by putting $y_A = -y_0$ and $\theta_A = -\theta_0$ into (18). This gives $P_0 = (2k/\lambda)y_0$ and $M_0 = (k/\lambda^3)\theta_0$, which, in turn, when applied to the infinite beam, furnish the solution for the case shown in Figure 19:

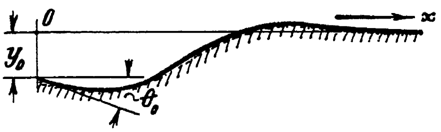

FIG. 19

$$\left.\begin{aligned} y &= y_0 A_{\lambda x} + \frac{1}{\lambda}\theta_0 B_{\lambda x} , \\ \theta &= -2\lambda y_0 B_{\lambda x} + \theta_0 C_{\lambda x} , \\ M &= 2\lambda EI(\lambda y_0 C_{\lambda x} + \theta_0 D_{\lambda x}), \\ Q &= -2\lambda^2 EI(2\lambda y_0 D_{\lambda x} + \theta_0 A_{\lambda x}), \end{aligned}\right\} \qquad (22\,\text{a–d})$$

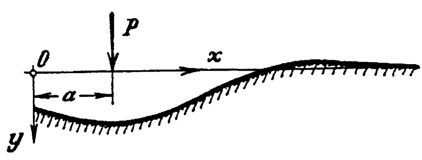

FIG. 20

In the same manner formulas can be derived also for situations in which the semi-infinite beam is loaded within the end $x = 0$. Taking, for instance, a semi-infinite beam loaded by a concentrated force P at a distance a from the free end, as shown in Figure 20, we can obtain the corresponding end-conditioning forces by putting, on the basis of (5c–d),

$$M_A = \frac{P}{4\lambda} C_{\lambda a} \quad \text{and} \quad Q_A = \frac{P}{2} D_{\lambda a}$$

into (16). This gives

$$P_0 = P(C_{\lambda a} + 2D_{\lambda a}) \quad \text{and} \quad M_0 = -\frac{P}{\lambda}(C_{\lambda a} + D_{\lambda a}),$$

which, when applied to the infinite beam at point O (at a distance a to the left of the force P), yields the equation of the deflection line as

$$y = \frac{P\lambda}{2k}[(C_{\lambda a} + 2D_{\lambda a})A_{\lambda x} - 2(C_{\lambda a} + D_{\lambda a})B_{\lambda x} + A_{\lambda|a-x|}]. \tag{23a}$$

If the notation $C_{\lambda a} + 2D_{\lambda a} = \alpha$ and $C_{\lambda a} + D_{\lambda a} = \beta$ is used, the expressions for slope, moment, and shearing force will take the form

$$\left.\begin{aligned} \theta &= -\frac{P\lambda^2}{k}(\alpha B_{\lambda x} + \beta C_{\lambda x} \pm B_{\lambda|a-x|}),\\ M &= \frac{P}{4\lambda}(\alpha C_{\lambda x} - 2\beta D_{\lambda x} + C_{\lambda|a-x|}),\\ Q &= -\frac{P}{2}(\alpha D_{\lambda x} - \beta A_{\lambda x} \pm D_{\lambda|a-x|}). \end{aligned}\right\} \tag{23 b–d}$$

The last term in each of the equations above represents the effect of the concentrated force P on the infinitely long beam. The a–x difference should be taken always in absolute value. In the last terms of (23 b and d) the upper, plus, sign refers to points with $x > a$, while the minus sign should be taken for points for which $x < a$.

Finally, let us determine the solution for a semi-infinite beam with free end subjected to distributed loading q, as shown in Figure 21. If the beam had infinite extension in both directions, at point O at the left end of the loaded portion l, according to (7 c–d), a bending moment $M = (q/4\lambda^2)B_{\lambda l}$ and a shearing force $Q = (q/4\lambda)(1 - C_{\lambda l})$ would be produced. The end-conditioning forces, which are to cancel this moment and shear at O, can be obtained by putting $M_A = (q/4\lambda^2)B_{\lambda l}$ and $Q_A = (q/4\lambda)(1 - C_{\lambda l})$ into (16), which gives then $P_0 = (q/\lambda)(1 + B_{\lambda l} - C_{\lambda l})$ and $M_0 = -(q/2\lambda^2)(1 + 2B_{\lambda l} - C_{\lambda l})$. Applying M_0 and Q_0 on the infinite beam simultaneously with the distributed loading q, we have the equation of the deflection line for abscissa values $0 < x < l$:

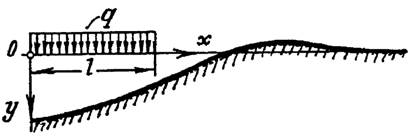

FIG. 21

$$y = \frac{q}{2k}[(1 + B_{\lambda l} - C_{\lambda l})A_{\lambda x} - (1 + 2B_{\lambda l} - C_{\lambda l})B_{\lambda x} + (2 - D_{\lambda x} - D_{\lambda(l-x)})]. \tag{24'}$$

At the same time for $x > l$ we have, according to (9a),

$$y = \frac{q}{2k}[(1 + B_{\lambda l} - C_{\lambda l})A_{\lambda x} - (1 + 2B_{\lambda l} - C_{\lambda l})B_{\lambda x} - (D_{\lambda x} - D_{\lambda(x-l)})]. \tag{24''}$$

Formulas for θ, M, and Q in the problem above can be obtained by successive differentiation of (24′) or (24″).

III. Applications

10. *The Railroad Track*

The theory of beams on elastic foundation found its first application in the calculation of stresses and deflections of railroad tracks.* Actually, in the modern crosstie systems only the ties are continuously supported by the roadbed, while the rail itself rests on the ties, that is, on closely spaced elastic supports. Investigations have shown, however, that an equivalent continuous elastic foundation can be substituted with good approximation for such supports, and in this way the theory can be applied to the analysis of the rails themselves.

If two equal forces N, corresponding to the rail pressures, are symmetrically applied to a tie, and they cause y_n deflection at their point of application, then the elasticity of the support, which is furnished by one tie to the rail, can be characterized by the

$$D = \frac{N}{y_n} \tag{a}$$

factor (spring constant). If such elastic supports are sufficiently closely spaced along the rail they can be replaced by a continuously distributed imaginary foundation,† the modulus of which is taken

$$k = \frac{D}{a}, \tag{b}$$

where a is the spacing of the ties and D is the spring constant to be determined experimentally. Introducing this modulus k into the previously derived formulas of the infinitely long beam, we can use those formulas for the analysis of rails.

Under a single wheel load P the maximum bending stress will then be, from (5c),

$$\sigma_{\max} = \frac{M_{\max}}{S} = \frac{P}{4\lambda S} = \frac{P}{4S}\sqrt[4]{\frac{4EI}{k}}, \tag{c}$$

where S denotes the section modulus of the rail. Introducing the cross-sectional area A of the rail into (c), we can write

$$\sigma_{\max} = \frac{P}{A}\,\frac{A\sqrt[4]{I}}{4S}\,\sqrt[4]{\frac{4E}{k}}. \tag{d}$$

The second term on the right side of the equation is constant for geometrically similar cross sections, and the third term does not depend on the dimensions of the rail. Hence we can conclude that the maximum bending stress in the rail

* E. Winkler, *Die Lehre von der Elastizität und Festigkeit* (Prag, 1867), pp. 182–184; and Zimmermann, *op. cit.*

† This approximation was introduced by S. Timoshenko in a paper on "Strength of Rails," *Transactions of the Institute of Ways of Communication* (St. Petersburg, 1915).

can be kept constant if with the increase of the wheel load P the cross-sectional area A (i.e. the weight of the rail) is also proportionally increased.*

It is seen from (c) that an error in the determination of the modulus k will not influence substantially the value of σ_{max}. Putting, for instance, $2k$ instead of k into (c), we find that the 100 per cent difference introduced in k causes only 16.5 per cent, that is, $(1 - 1/\sqrt[4]{2})$, deviation in the value of the maximum bending stress.

The approximation achieved by substituting a continuous elastic support for the separate ones is not restricted to the analysis of the railroad track, but may find application to various engineering structures. As a rule, separate elastic supports may be replaced by an imaginary continuous foundation if we have at least four of the supports in the characteristic wave length $l = \pi/\lambda$ of the deflection line. This means that for such cases we must satisfy the formula

$$\sqrt[4]{\frac{64a^2D}{EI}} < \pi, \tag{e}$$

where a denotes the spacing and D the elasticity of the supports, and where EI indicates the flexural rigidity of the supported longitudinal beam.

Figure 22† shows the moment diagram of a beam of unlimited length subjected to a single concentrated force and analyzed, first, as resting on separated elastic supports (polygonal moment diagram) and, secondly, as supported on an equiva-

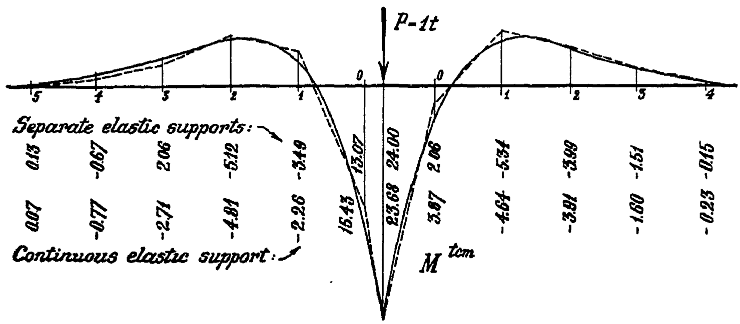

FIG. 22

lent elastic foundation (continuous moment curve). The agreement between the deflection lines, calculated by the two different assumptions, is a close one and the differences are quite insignificant.

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* See the paper by S. Timoshenko and B. F. Langer, "Stresses in Railroad Track," *Transactions of the American Society of Mechanical Engineers*, vol. 54 (1932), Applied Mechanics Section, pp. 277–302.

† Figure 22 was taken from A. Wasiutyński, *Recherches expérimentales sur les déformations élastiques et le travail de la superstructure des chemins de fer*, in *Annales de l'Academie des Sciences techniques à Varsovie*, Tome IV (1937).

In the case of a railroad track the ballast and especially the underlying subsoil represent a certain continuity between the deflection of neighboring points. Such continuity was not considered in the fundamental assumption (equation [a], p. 2) on which the present theory is based, but its influence can be taken intc account if the value of D in (b) above is determined in the following experimental way.

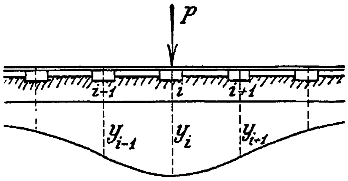

FIG. 23

Suppose that the rail is subjected to a wheel load P and the deflection ordinates of the rail are measured at every tie (Fig. 23). Assuming that the spring constant D is the same for every tie, from the consideration of equilibrium we get $P = D \sum y_i$ and hence

$$D = \frac{P}{\sum y_i}. \tag{f}$$

Such experiments have been done recently by A. Wasiutyński (*op. cit.*), who points out that the value of D determined by means of (f) is about half the value obtained when only a single separate tie is loaded (equation [a], p. 27). The reason for this difference is that on account of the continuity in the subsoil the deflection of the neighboring ties to the right and left of point i facilitates the deflection of the ith tie, and, in consequence of this, one tie in such an assembly will be about twice as flexible (half as large as D) as a separate single tie under the loading. The value of D determined from (f) corresponds to the situation actually existing in the railroad track. Putting this D into (b) and introducing the modulus k so obtained into the formulas for the infinitely long beam, we find

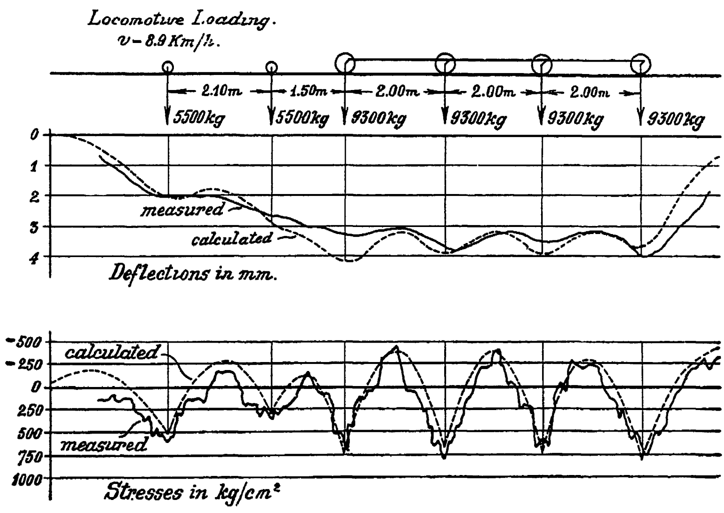

FIG. 24

the calculated results to be in remarkable agreement with the actual stress and deflection measurements. Such a comparison, taken from the work of Wasiutyński cited on page 28, is shown in Figure 24. One can observe that the stress measurements usually check more closely with the theoretical results than do the deflection measurements. This might be explained by the fact that an inaccuracy in the determination of k has greater effect on the deflection line than on the moment curve. While a 100 per cent difference in the value of k would cause only 16.5 per cent deviation in σ_{max} (as shown on p. 28), it would produce, from equation (5a), a 40 per cent difference, that is, $(1 - \frac{1}{2}\sqrt[4]{2})$, in the value of the maximum deflection ordinate of the beam under a single concentrated force.

The elements of a normal track have in general the following dimensions: ties, length 8–9 ft., cross section 8 × 6–9 × 7 in., spacing 24–30 in. on centers; ballast (crushed stone), average depth 14 in. Experiments carried out on such tracks (under simultaneous loading of several ties) gave for the elastic coefficients introduced above the average values

$$D = 48{,}000 \text{ lbs./in.} - 60{,}000 \text{ lbs./in.}$$

and

$$k = 1400 \text{ lbs./in.}^2 - 2000 \text{ lbs./in.}^2 *$$

From such measurements also the k_0 modulus of the foundation of the ties can be derived. This k_0 is found to have a value (when several neighboring ties are simultaneously loaded) of $k_0 = 110 - 130$ lbs./in.^{2} per inch.

11. *Cylindrical Tube under Axially Symmetrical Loading*

Consider a thin-walled cylindrical tube subjected to radial forces uniformly distributed along an arbitrary circle on the tube (Fig. 25a).

Because of the symmetry of such loading every section normal to the axis will remain circular, while the radius R will undergo a change, $\Delta R = y$, different for each cross section. The radial displacement y can be regarded as deflection for a longitudinal element of the tube, and hence it is seen that the assumed loading will set up bending stresses in the longitudinal elements. On account of the symmetry, we have to consider the deformation of only one element. Let it be A–B, shown in Figure 25a, and assume its width equal to unity.

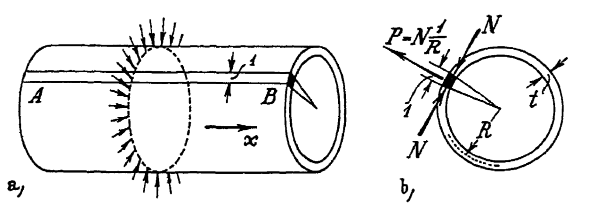

FIG. 25

* For further details see the Progress Reports of the Special Committee on Stresses in Railroad Track (under the direction of A. N. Talbot), published in the *Transactions of the American Society of Civil Engineers*, Vols. 82–83, 86, 88 (1918, 1920, 1923, 1925); and in the *American Railway Engineering Association Bulletin*, Vol. 31, No. 319 (1929).

The radial displacement y must be accompanied by a circumferential compression y/R of the tube, which in turn will give rise to compressive hoop forces N (Fig. 25b) having a magnitude of

$$N = \frac{Et}{R} y \tag{a}$$

per unit length of the A–B element, R being the middle radius and t the thickness of the tube. The resultant of these forces N will have a radial direction and be of the value

$$P = N \frac{1}{R} = \frac{Et}{R^2} y. \tag{b}$$

It is seen that this force P, opposing the deflection, will also be proportional to the deflection, Et/R^2 being the proportionality factor. Hence we may conclude that a longitudinal element of a cylindrical tube loaded symmetrically with respect to its axis can be regarded as a beam on an elastic foundation, the modulus of which,

$$k = \frac{Et}{R^2}, \tag{25}$$

depends on the cross-sectional dimensions and on the material of the tube. The width of the longitudinal element in the derivation above was taken equal to unity.

When dealing with the bending of a longitudinal element we must consider that, on account of the axial symmetry of the deformation of the tube, the sides of each longitudinal element will not be able to rotate in order to permit the lateral extension or compression of the longitudinal fibers, but will have to remain parallel with their original (radial) directions. This restraining influence is equivalent to a bending moment

$$M_c = \mu M_x , \tag{c}$$

in the circumferential ring where M_x denotes the bending moment in the longitudinal beam and μ is Poisson's ratio for the material. The stiffening effect of M_c on the bending deformation of the longitudinal beams can be taken into account by increasing the moment of inertia of each beam of unit width in the ratio $1/(1 - \mu^2)$, and thus we have

$$I = \frac{t^3}{12(1 - \mu^2)}. \tag{d}$$

Using the expressions above for k and I, we shall obtain the characteristic λ for the tube as

$$\lambda = \sqrt[4]{\frac{k}{4EI}} = \sqrt[4]{3(1 - \mu^2)} \cdot \frac{1}{\sqrt{Rt}}. \tag{26}$$

If steel ($\mu = 0.3$) is the material of the tube we have

$$\lambda = 1.285 \frac{1}{\sqrt{Rt}}. \tag{e}$$

Since any loading which is symmetrical with respect to the axis of the tube can be regarded as composed of such concentrated elements of loading as that shown in Figure 25a, from the principle of superposition it follows that the same method may be used equally well in the general case when the load varies in an arbitrary manner along the axis of the cylinder, provided that in each normal section the radial loading is uniformly distributed along the circumference of the cross section, which means that the axial symmetry of the loading is preserved. In problems of this type the deformation and stress in the longitudinal elements can be obtained from the elastically supported beam formulas as y_x, θ_x, M_x, and Q_x, respectively, by taking k and λ as in (25) and (26). At the same time the stress in the circumferential direction is obtained by adding to the normal hoop stress, σ_n, the stress caused by the circumferential bending moment, σ_b. Thus we have as extreme fiber stress in the circumferential direction

$$\sigma_c = \sigma_n + \sigma_b = -\frac{E}{R} y_x \pm \frac{6}{t^2} \mu M_x, \tag{27}$$

where the plus and minus signs on the right side give the fiber stresses at the inside and at the outside surface of the tube respectively, tensile stress always being regarded as positive.

If we assume a thin-walled cylinder without external load to be subjected to a change of temperature, constant in any of the cross sections but varying arbitrarily along the axis of the cylinder, we have again a problem of axially symmetrical deformation to which the theory of beams on elastic foundation applies.

Let T_0 denote the initial uniform temperature of the tube and T the temperature after heating has taken place, assuming the latter to be uniform in each cross section and to vary only along the axis of the tube. The radial displacement y of the element A–B (Fig. 25a) will be accompanied by a unit strain y/R in the circumferential direction; this strain is due partly to the circumferential compressive forces N and partly to the temperature change $(T - T_0)$ in the cross section under consideration. Denoting by α the coefficient of linear expansion for the material of the tube, we have

$$\frac{y}{R} = \frac{N}{Et} - \alpha(T - T_0), \tag{f}$$

from which

$$N = \frac{Et}{R} y + \alpha Et(T - T_0). \tag{28}$$

The corresponding radial forces acting on the element A–B will be

$$p = N\frac{1}{R} = \frac{Et}{R^2} y + \alpha \frac{Et}{R}(T - T_0) \tag{g}$$

and will be equal to $-\frac{EI}{1-\mu^2}\frac{d^4y}{dx^4}$, as is known from the bending theory of beams. Hence we have the differential equation for our problem

$$\frac{EI}{1-\mu^2}\frac{d^4y}{dx^4} = -\left[\frac{Et}{R^2}y + \alpha\frac{Et}{R}(T - T_0)\right]. \tag{29}$$

Comparing this expression with equation (1), on page 3, we find that the second term on the right side of the equation above corresponds to a distributed loading $-q$ acting on a beam of flexural rigidity $EI/(1 - \mu^2)$ supported on an elastic foundation with the modulus $k = Et/R^2$. Hence the distribution of thermal stresses in a cylindrical tube due to a variation of temperature $(T - T_0)$ along the axis of the tube can be determined by simply taking $\alpha(Et/R)(T - T_0)$ as a distributed load on the tube.* Any of the previously derived formulas for beams on elastic foundation will then give the solution for deflections, moments, and shears in the longitudinal elements of the tube, while the circumferential forces N will be determined by (28).

12. *Examples*

1. Because of the wavy course of the moment influence line on an infinite beam (Fig. 5c), it is possible to place on the beam a number of equal forces P in such a manner that the resulting maximum bending stress will be less than that caused by a single force P alone.

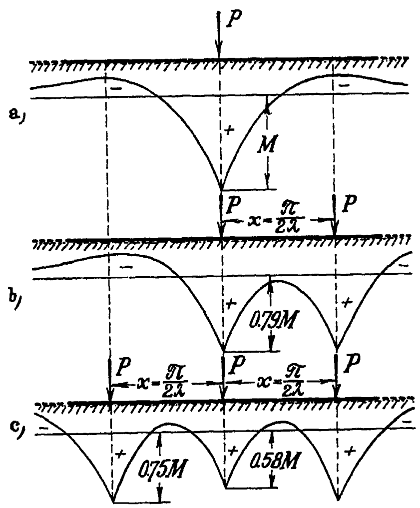

FIG. 26

A single force P on the infinite beam will produce a maximum bending moment, from (5c), equal to $M = P/4\lambda$ (Fig. 26a). From the influence line for moment shown in Figure 5c we see that another force, P, applied at a distance $x = \pi/2\lambda$, will reduce the former maximum moment to

$$M(1 + C_{\lambda x=\pi/2}) = M(1 - 0.2079) = 0.7921\,M \quad \text{(Fig. 26b)}.$$

If one more P load is added to the previous two with the same spacing, $x = \pi/2\lambda$, the bending moment under the center load will be

$$M(1 + 2C_{\lambda x=\pi/2}) = M(1 - 0.4158) = 0.5842\,M,$$

while under the side loads we shall have the value

$$M(1 + C_{\lambda x=\pi/2} + C_{\lambda x=\pi}) = M(1 - 0.2079 - 0.0432) = 0.7489\,M \quad \text{(Fig. 26c)}.$$

* See S. Timoshenko, *Strength of Materials* (2d ed.; New York, 1941), Part II, p. 174; and also J. P. den Hartog, "Temperature Stresses in Flat Rectangular Plates and Thir Cylindrical Tubes," *Journal of the Franklin Institute*, 222 (1936), 149–181.

2. Consider an infinite beam loaded as shown in Figure 27. Let us set the problem of finding for the beam the value of I which gives such a pressure distribution in the foundation that the maximum pressure at the center of the loaded portion will equal $\frac{1}{4}q$. Using (7a), we can write this requirement as

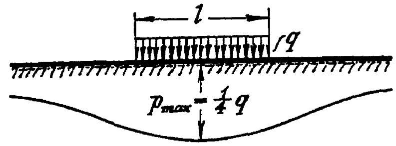

FIG. 27

$$p_{\max} = q\left(1 - e^{-\lambda l/2}\cos\frac{\lambda l}{2}\right) = \tfrac{1}{4}q,$$

or, putting $x = \frac{1}{2}\lambda l$ and $e^{-x}\cos x = D_x$, we find that the condition must fulfil[l] the equation

$$D_x - 0.75 = 0.$$

Such equations can be solved most easily by Newton's approximation.*

As a first approximation for x we have from the tables (p. 221) $x_1 = 0.25$, which gives

$$f(x_1) = D_{x_1} - 0.75 = 0.7546 - 0.7500 = 0.0046$$

and

$$f'(x_1) = \left[\frac{d}{dx}D_x\right]_{x=x_1} = -A_{x_1} = -0.9472.$$

Hence the second approximation for x will be

$$x_2 = 0.2500 + \frac{0.0046}{0.9472} = 0.25485.$$

Since

$$x = \tfrac{1}{2}\lambda l = 0.25485,$$

from $\lambda = \sqrt[4]{k/4EI} = (2/l)0.25485$ we have the expression for the required moment of inertia as

$$I = 3.704\frac{kl^4}{E}.$$

* If x_1 denotes the first approximation of the root of equation $f(x) = 0$, a second improved value for the root can be obtained as

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

This step can be repeated until the required accuracy in the value of x is secured. This method is especially adapted to solving equations involving the A, B, C, and D functions, since they are derivatives of one another (see the equations on p. 12), and consequently $f(x_1)$ and $f'(x_1)$ can be taken out of the same table.

3. A beam of flexural rigidity EI is supported on the free ends of a series of cantilevers, each having bending stiffness E_0I_0, and loaded with a distributed load q (Fig. 28). Assume that the spacing c of the cantilevers satisfies the condition (e) on page 28 and the longitudinal beam can be regarded as one supported on a continuous elastic foundation, the modulus of which will here be $k = D/c = 3E_0I_0/ca^3$. If the flexural rigidity of one of the cantilever beams is increased, a concentrated reaction force R will be produced at that place, causing bending in the longitudinal girder. Let us find the value of R produced if the value of E_0I_0 for one of the cantilevers is doubled.

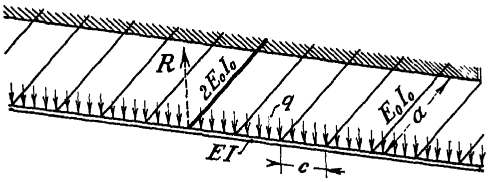

FIG. 28

If all cantilever beams had the same E_0I_0 value, the deflection of the main girder would be constant, $y_0 = q/k = q(ca^3/3E_0I_0)$. If the flexural rigidity of one of the cantilevers is doubled the deflections at that point must satisfy the requirement

$$y_0 = R\left(\frac{\lambda}{2k} + \frac{1}{kc}\right),$$

from which we have

$$R = q\frac{2c}{\lambda c + 2},$$

where $\lambda = \sqrt[4]{k/4EI}$ and k is as given above.

4. If a circular tube of unlimited length is subjected to an internal pressure p, we find, from considerations of statics, that there will be produced in the wall of the tube a uniform circumferential stress $\sigma_c = pR/t$, where R denotes the middle radius and t the wall thickness of the tube. Because of this stress the tube will expand radially by the amount $\delta = pR^2/Et$. Suppose that at one place a so-called reinforcing ring is applied to the tube which prevents the expansion at its point of application (Fig. 29). The effect of this ring will be the same as the effect of a concentrated force producing deflection $y = pR^2/Et$ on an infinite beam supported on an elastic foundation having the values of k and λ defined by (25) and (26). The maximum bending moment due to this action can be determined from (5 a and c), and so we get (using $\mu = 0.3$) for the maximum bending stress

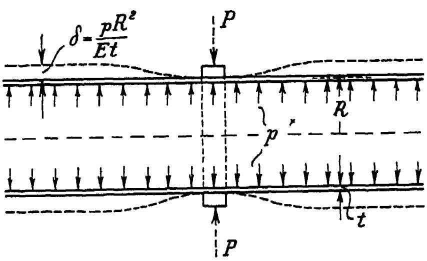

FIG. 29

$$\sigma_{\max} = \frac{6M_{\max}}{t^2} = 1.82\frac{pR}{t}.$$

It is seen that this bending stress, which occurs under the reinforcing ring in the longitudinal fibers of the tube, is 82 per cent larger than the circumferential stress existing before the application of the ring.*

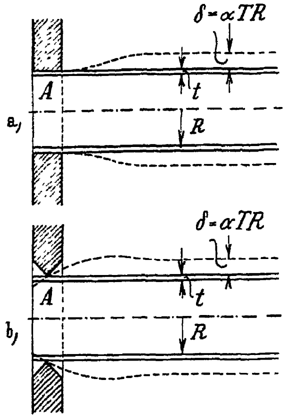

FIG. 30

5. Consider a circular tube with one end built into a wall made of the same material as the tube (Fig. 30a) and assume that the temperature of the tube is raised by T degrees with respect to the temperature of the wall. By the fixing of the end A a radial extension $\delta = \alpha TR$ of the tube is prevented, α denoting the coefficient of thermal expansion of the material and R the radius of the tube. At the same time, owing to the fixed end, the slope of the deformation curve at A must also be equal to zero. Putting $y_0 = \alpha TR$ and $\theta_0 = 0$ into (22c), we obtain the maximum bending moment at A ($x = 0$) as

$$M_{\max} = 2\alpha TREI\lambda^2.$$

Since for the tube

$$I = \frac{t^3}{12(1 - \mu^2)} \quad \text{and} \quad \lambda^2 = \sqrt{3(1 - \mu^2)}\,\frac{1}{Rt},$$

we can, by using $\mu = 0.3$, write the expression for the maximum bending moment as

$$M_{\max} = 0.303\alpha TEt^2.$$

If we assume such fixing at A as permits a rotation of the end (Fig. 30b), an expression for the M curve can be obtained by taking $y_0 = \alpha TR$ and $[M]_{x=0} = 0$ in (22c). This gives $\theta_0 = -\lambda y_0$, and now we get, using the expressions above for I and λ, the equation of the bending-moment diagram as $M = -0.303\alpha TEt^2e^{-\lambda x}\sin\lambda x$, the maximum value of which, at a distance $x = \pi/4\lambda$ from point A, will be

$$M_{\max} = -0.098\alpha TEt^2.$$

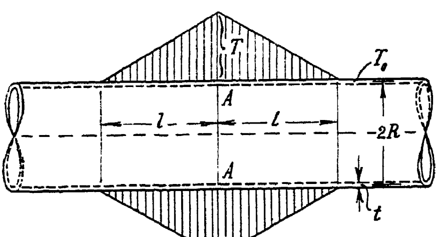

FIG. 31

6. Consider a cylindrical tube of unlimited extension heated along a circumferential circle to a temperature of T degrees. Assume that from point A of the heated circle the temperature decreases on both sides linearly along a length l until it reaches the uniform initial temperature T_0 of the tube (Fig. 31). Such a condition can occur when two tubes are butt-joined by welding.

* In spite of their stress-raising effect, reinforcing rings are frequently used for constructional reasons, such as to increase the stiffness of the tube, to provide support, and so on.

The problem is to find the thermal stresses set up in the tube. The solution can be obtained if, according to (29), we subject the tube to a triangular loading whose maximum ordinate is equivalent to

$$q_0 = -\alpha \frac{Et}{R} (T - T_0).$$

Substituting this q_0 value in (10c) and putting $\lambda = \sqrt[4]{3(1 - \mu^2)}(1/\sqrt{Rt})$, $a = l$, $b = 0$, we have the maximum bending moment occurring at A (the value given by [10c] is doubled because of the double triangular load):

$$M_A = \frac{0.110}{(1 - \mu^2)^{3/4}} \alpha(T - T_0)E\sqrt{Rt^5} \frac{A_{\lambda l} - 1}{l};$$

the maximum longitudinal bending stress will be (if $\mu = 0.3$)

$$\sigma_{\max} = 0.707\alpha(T - T_0)E\sqrt{Rt} \frac{A_{\lambda l} - 1}{l}.$$

The deflection at point A is obtained from (10a), with $k = Et/R^2$, as

$$y_A = -0.389\alpha(T - T_0)\sqrt{R^3t} \frac{C_{\lambda l} + 2\lambda l - 1}{l};$$

and for the normal hoop stress σ_n, after substituting in (28) the y_A value above, we have

$$\sigma_n = \frac{N_A}{t} = \alpha(T - T_0)E\left(1 - 0.389\sqrt{Rt} \frac{C_{\lambda l} + 2\lambda l - 1}{l}\right).$$

This normal stress is acting in the middle plane of the wall of the tube. The extreme fiber stress in the circumferential direction is obtained (according to [27]) by adding to the σ_n value above the stress produced by bending in the circumferential direction:

$$\sigma_b = 6\mu M_A/t^2.$$

CHAPTER III

BEAMS OF FINITE LENGTH

13. *General Method of Solution for Beams of Finite Length*

For a beam of finite length the correct solution is the one which, besides fulfilling the differential equation of the elastic line, also satisfies the required conditions at both ends of the beam.

The previously derived elastic curves of the infinite beam all satisfy the differential equation of bending. In consequence of the principle of superposition any combination of particular loadings will also satisfy that differential equation. Hence it follows that, if we find such a combination of loadings as fulfills the conditions prescribed for certain points (end points) of the elastic line, we shall thus obtain the solution for a beam of finite length.

Assume that an infinite beam is subjected to a given loading P and q, as shown in Figure 32. Because of this loading certain values of y, θ, M, and Q will be produced at points A and B. By superposing on this loaded beam two pairs of concentrated forces and moments (M_{0A}, P_{0A} and M_{0B}, P_{0B}) we can modify the elastic curve in such a way that at points A and B the required end conditions will be fulfilled. For each end (A and B) we can prescribe two conditions.

Since the four quantities P_{0A}, M_{0A} and P_{0B}, M_{0B} *create* the required conditions at A and B, they will be collectively termed, in the discussion which follows, the *end-conditioning forces*. This end-conditioning principle was used in the preceding chapter when we derived solutions for the semi-infinite beams. The present use differs only in that now we must consider also the influence of these end-conditioning forces on each other, for P_{0A} and M_{0A} will have an effect on the conditions existing at B, and similarly P_{0B} and M_{0B} will have an influence at point A. Thus the correct value for these end-conditioning forces must be determined from four simultaneous equations, representing the simultaneous fulfillment of the end conditions at points A and B. It will be assumed that these end-conditioning forces are applied infinitely close to the *outer side* of the A–B portion. In this way any uncertainty which might arise from the singular character of the points of application will be eliminated. The assumed positive sense for the end-conditioning forces is shown in Figure 32. The procedure for finding their values in any particular case of end conditions will be discussed in the following sections.

14. *Beams with Free Ends*

Consider an infinitely long beam subjected to a given loading, as shown in Figure 33a. Our aim will be to obtain from it a solution for the beam of finite length which is under the same loading and has free ends at A and B (Fig. 33c). In the beam in Figure 33a there are bending moments and shearing forces produced at A and B, but in that in Figure 33c we have $M = 0$ and $Q = 0$ at those points. Our method will be to make the moments and shears vanish at points A and B on the infinite beam, by applying the end-conditioning forces P_{0A}, M_{0A} and P_{0B}, M_{0B} at those points. According to the reasoning previously presented, the infinitely long bar under the action of the given loading and the end-conditioning forces (Fig. 33b) will behave between points A and B exactly in the same manner as if it had free ends at these points (Fig. 33c).

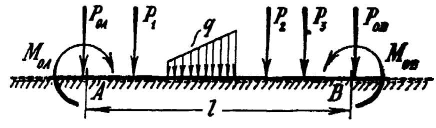

FIG. 32

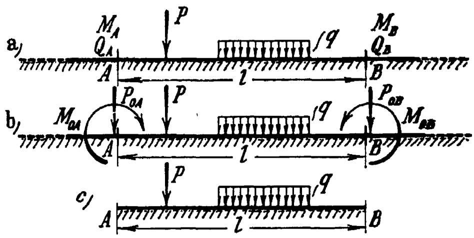

FIG. 33

The end-conditioning forces must produce $-M_A$, $-Q_A$, $-M_B$ and $-Q_B$ at the points A and B. Having altogether four conditions for the two ends, we can determine the value of the four end-conditioning forces.

By use of (5 c–d) and (6 c–d), the conditions above for free ends can be written as

$$\left.\begin{aligned}
M_A + \frac{P_{0A}}{4\lambda} + \frac{P_{0B}}{4\lambda} C_{\lambda l} + \frac{M_{0A}}{2} + \frac{M_{0B}}{2} D_{\lambda l} &= 0,\\
Q_A - \frac{P_{0A}}{2} + \frac{P_{0B}}{2} D_{\lambda l} - \frac{\lambda M_{0A}}{2} + \frac{\lambda M_{0B}}{2} A_{\lambda l} &= 0,\\
M_B + \frac{P_{0A}}{4\lambda} C_{\lambda l} + \frac{P_{0B}}{4\lambda} + \frac{M_{0A}}{2} D_{\lambda l} + \frac{M_{0B}}{2} &= 0,\\
Q_B - \frac{P_{0A}}{2} D_{\lambda l} + \frac{P_{0B}}{2} - \frac{\lambda M_{0A}}{2} A_{\lambda l} + \frac{\lambda M_{0B}}{2} &= 0.
\end{aligned}\right\} \quad (30)$$

By solving such a system of equations we can determine the values of the end-conditioning forces in each case, but an explicit form for the unknowns in (30) would lead to complicated expressions.

In order to simplify the solution of these equations we shall resolve the original loading (as a simple example consider that shown in Fig. 34a) into two parts, a *symmetrical* part (Fig. 34b) and an *antisymmetrical one** (Fig. 34c). By resolving in this way any given loading into symmetrical and antisymmetrical

* This method is sometimes referred to in the literature as the "Andree–Herzka method." See W. L. Andree, *Zur Berechnung statisch unbestimmter Systeme, Das B–U Verfahren* (München–Berlin, 1919).

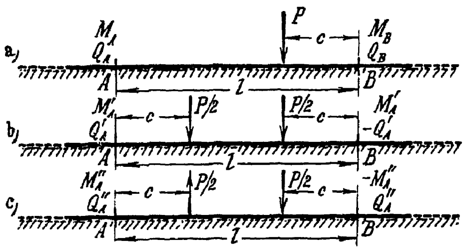

FIG. 34

components we actually resolve the system of four equations shown in (30) into two systems with two unknowns in each. Thus in Figure 34b, as well as in Figure 34c, the moments and shears at the end points A and B are equal in absolute value and differ only in signs, as is shown there. Hence in each of these situations the end-conditioning forces will be of equal value too, and their calculation will need, in each instance, the solution of only two simultaneous equations.

Since a considerable simplification in the computing work is attainable by this procedure, we shall make extensive use of it in the future. The consecutive steps in applying the method will be: (*a*) resolving the original loading into a symmetrical and an antisymmetrical part, (*b*) determining the end-conditioning forces in each of these parts, and (*c*) adding them together afterward at each end point. By this means we obtain the total value of the end-conditioning forces for the original given loading.

The sign convention will have considerable importance in this procedure, and so we shall decide now* that M_A, Q_A and M_B, Q_B will denote the moments and shears at points A and B due to the original loading (Fig. 34a); that M'_A, Q'_A will represent the moment and shear at A due to the symmetrical loading (at B the values are the same; only the sign of the shear is reversed, as shown in Fig. 34b); and that M''_A, Q''_A will indicate the moment and shear at A due to the antisymmetrical loading (at B they are of the same value, but the sign of the moment is reversed; see Fig. 34c). Any of these symbols may denote negative quantities. For example, if when computing the shearing force at A this happens to come out negative, then Q_A will indicate a negative quantity.

According to this notation, it follows from Figure 34 a–c that

$$M_A = M'_A + M''_A, \qquad M_B = M'_A - M''_A,$$

$$Q_A = Q'_A + Q''_A, \qquad Q_B = -Q'_A + Q''_A,$$

and, therefore,

$$\left.\begin{aligned} M'_A &= \tfrac{1}{2}(M_A + M_B), \qquad & Q'_A &= \tfrac{1}{2}(Q_A - Q_B), \\ M''_A &= \tfrac{1}{2}(M_A - M_B), \qquad & Q''_A &= \tfrac{1}{2}(Q_A + Q_B). \end{aligned}\right\} \tag{31}$$

Having determined the M_A, M_B, Q_A, Q_B values due to the original loading on the infinite beam, we can find from (31) the corresponding M'_A, Q'_A and M''_A, Q''_A values for the symmetrical and antisymmetrical loadings respectively.

* In addition to the general conventions established on p. 11.

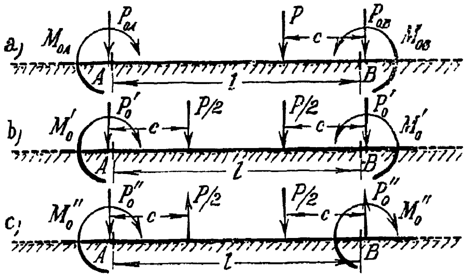

FIG. 35

The next step will be to remove moments and shearing forces at points A and B by applying at these points P_0', M_0' in the symmetrical case and P_0'', M_0'' in the antisymmetrical case, as shown in Figure 35 b–c. Both these figures indicate the assumed positive directions for the end-conditioning forces. By superposing the symmetrical and antisymmetrical components we shall obtain the end-conditioning forces for the original case (Fig. 35a), in which

$$P_{0A} = P_0' + P_0'', \qquad P_{0B} = P_0' - P_0'',$$
$$M_{0A} = M_0' + M_0'', \qquad M_{0B} = M_0' - M_0''.$$

In order to remove the moments and shears at A and B on the infinite beam we shall require that P_{0A}, M_{0A}, P_{0B}, and M_{0B} should produce altogether $-M_A$, $-Q_A$ at A and $-M_B$, $-Q_B$ at B, which they will do if: P_0' and M_0' produce

$$-M_A', \; -Q_A' \text{ at } A \quad \text{and} \quad -M_A', \; +Q_A' \text{ at } B, \tag{a}$$

and if P_0'' and M_0'' produce

$$-M_A'', \; -Q_A'' \text{ at } A \quad \text{and} \quad +M_A'', \; -Q_A'' \text{ at } B. \tag{b}$$

From conditions (a) and (b) above the values for the end-conditioning forces can be determined separately for the symmetrical and the antisymmetrical cases:

a. Symmetrical Case

Using (5 c–d) and (6 c–d), we can write the conditions in (a) above as

$$\left.\begin{aligned} \frac{P_0'}{4\lambda}(1 + C_{\lambda l}) + \frac{M_0'}{2}(1 + D_{\lambda l}) &= -M_A', \\ -\frac{P_0'}{2}(1 - D_{\lambda l}) - \frac{\lambda M_0'}{2}(1 - A_{\lambda l}) &= -Q_A'. \end{aligned}\right\} \tag{c}$$

From this we have

$$\left.\begin{aligned} P_0' &= 4E_I[Q_A'(1 + D_{\lambda l}) + \lambda M_A'(1 - A_{\lambda l})], \\ M_0' &= -\frac{2}{\lambda}E_I[Q_A'(1 + C_{\lambda l}) + 2\lambda M_A'(1 - D_{\lambda l})], \end{aligned}\right\} \tag{32'}$$

where

$$E_I = \frac{1}{2(1 + D_{\lambda l})(1 - D_{\lambda l}) - (1 - A_{\lambda l})(1 + C_{\lambda l})}.^*$$

---

* In another form:

$$E_I = \frac{1}{2}\frac{e^{\lambda l}}{\text{Sinh}\,\lambda l + \sin\lambda l}, \quad E_{II} = \frac{1}{2}\frac{e^{\lambda l}}{\text{Sinh}\,\lambda l - \sin\lambda l}.$$

b. Antisymmetrical Case

Using (5 c–d) and (6 c–d), we can put the conditions in (b) in the form

$$\left.\begin{aligned} \frac{P_0''}{4\lambda}(1 - C_{\lambda l}) + \frac{M_0''}{2}(1 - D_{\lambda l}) &= -M_A'', \\ -\frac{P_0''}{2}(1 + D_{\lambda l}) - \frac{\lambda M_0''}{2}(1 + A_{\lambda l}) &= -Q_A'', \end{aligned}\right\} \quad \text{(d)}$$

from which

$$\left.\begin{aligned} P_0'' &= 4E_{II}\,[Q_A''(1 - D_{\lambda l}) + \lambda M_A''(1 + A_{\lambda l})], \\ M_0'' &= -\frac{2}{\lambda}E_{II}\,[Q_A''(1 - C_{\lambda l}) + 2\lambda M_A''(1 + D_{\lambda l})], \end{aligned}\right\} \quad (32'')$$

where

$$E_{II} = \frac{1}{2(1 + D_{\lambda l})(1 - D_{\lambda l}) - (1 + A_{\lambda l})(1 - C_{\lambda l})}.^*$$

At the end of this book a table of E functions is given. The E quantities as well as the other multiplicators of the Q's and M's on the right side of (32′) and (32″) are functions of λl only; consequently, these coefficients have to be established only once for each beam and can be used for determining the end-conditioning forces for any type of loading.

If (32′) and (32″) give a negative value for an end-conditioning force, it will mean that the direction of that force should be opposite to the one that was assumed as positive (Fig. 35 b–c). In any particular problem it is advisable to show in a sketch the way in which these end-conditioning forces are acting, separately in the symmetrical and the anti-symmetrical cases, in order to avoid any mistake in signs when adding them together at the end points A and B respectively.

This procedure gives the solution for a beam of finite length subjected to any lateral loading. It includes also the particular situation when forces are acting only at the ends of the beam. From (32′) and (32″) it follows that P_0' and M_0' on the infinite beam (Fig. 36a) will create a situation between points A and B which is identical with that in which the beam of finite length A–B is acted upon at the ends by $-M_A'$, $-Q_A'$ and $-M_A'$, $+Q_A'$ end loads (Fig. 36b). Simi-

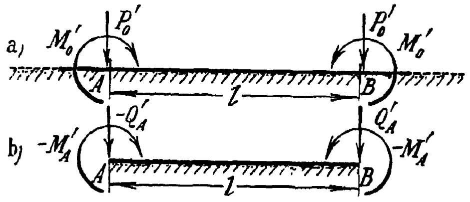

Fig. 36

* See note on page 41.

larly, Figure 37 a–b gives an identical situation on the A–B portion for the antisymmetrical case. In (32′) and (32″) we can assign any value to the quantities M_A, Q_A; consequently, in this way we can derive the solution for a beam of finite length subjected to any end loading.

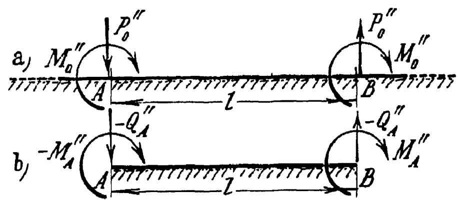

FIG. 37

15. *Beams with Hinged Ends*

For a beam with hinged ends, such as that shown in Figure 38, the conditions for both ends are

$$y = 0 \quad \text{and} \quad M = 0.$$

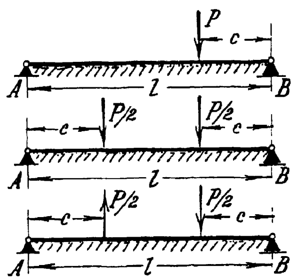

FIG. 38

Extending the sign convention previously adopted (p. 40) for deflections and slopes, we shall denote by y_A, θ_A, y_B, θ_B the deflections and slopes produced by the loading at points A and B respectively on an infinite beam; by y'_A, θ'_A the deflection and slope in the symmetrical case at A (at B the values remain the same, but the sign of θ is reversed); and by y''_A, θ''_A the deflection and slope in the antisymmetrical case at A (at B they are of the same value, but the sign of y is reversed). As was pointed out before, any of these symbols may denote a negative quantity. From the notations above it follows that

$$y_A = y'_A + y''_A, \qquad M_A = M'_A + M''_A,$$
$$y_B = y'_A - y''_A, \qquad M_B = M'_A - M''_A,$$

and, therefore,

$$\left.\begin{aligned} y'_A &= \tfrac{1}{2}(y_A + y_B), & M'_A &= \tfrac{1}{2}(M_A + M_B), \\ y''_A &= \tfrac{1}{2}(y_A - y_B), & M''_A &= \tfrac{1}{2}(M_A - M_B). \end{aligned}\right\} \tag{33}$$

We shall again split the original loading into a symmetrical and an antisymmetrical one and shall determine the end-conditioning forces separately for each.

a. Symmetrical Case

P'_0 and M'_0 must produce $-y'_A$ and $-M'_A$ at points A and B on the infinite beam. That is, by (5 a and c) and (6 a and c),

$$\left.\begin{aligned} \frac{P'_0\lambda}{2k}(1 + A_{\lambda l}) + \frac{M'_0\lambda^2}{k}B_{\lambda l} &= -y'_A, \\ \frac{P'_0}{4\lambda}(1 + C_{\lambda l}) + \frac{M'_0}{2}(1 + D_{\lambda l}) &= -M'_A. \end{aligned}\right\} \tag{a}$$

From this we have

$$\left.\begin{aligned} P_0' &= 4\lambda F_I[M_A' B_{\lambda l} - 2\lambda^2 EI y_A'(1 + D_{\lambda l})], \\ M_0' &= 2F_I[-M_A'(1 + A_{\lambda l}) + 2\lambda^2 EI y_A'(1 + C_{\lambda l})], \end{aligned}\right\} \qquad (34')$$

where

$$F_I = -\frac{1}{B_{\lambda l}(1 + C_{\lambda l}) - (1 + D_{\lambda l})(1 + A_{\lambda l})}.^*$$

b. *Antisymmetrical Case*

P_0'' and M_0'' must produce $-y_A''$ and $-M_A''$ at A and $+y_A''$ and $+M_A''$ at B. Using (5 a and c) and (6 a and c), we can write the conditions above as

$$\left.\begin{aligned} \frac{P_0''\lambda}{2k}(1 - A_{\lambda l}) - \frac{M_0''\lambda^2}{k} B_{\lambda l} &= -y_A'', \\ \frac{P_0''}{4\lambda}(1 - C_{\lambda l}) + \frac{M_0''}{2}(1 - D_{\lambda l}) &= -M_A'', \end{aligned}\right\} \qquad (b)$$

from which

$$\left.\begin{aligned} P_0'' &= -4\lambda F_{II}[M_A'' B_{\lambda l} + 2\lambda^2 EI y_A''(1 - D_{\lambda l})], \\ M_0'' &= -2F_{II}[M_A''(1 - A_{\lambda l}) - 2\lambda^2 EI y_A''(1 - C_{\lambda l})], \end{aligned}\right\} \qquad (34'')$$

where

$$F_{II} = \frac{1}{B_{\lambda l}(1 - C_{\lambda l}) + (1 - D_{\lambda l})(1 - A_{\lambda l})},^*$$

16. *Beams with Fixed Ends*

For a beam with fixed ends, such as that shown in Figure 39, the conditions for both ends are

$$y = 0 \quad \text{and} \quad \theta = 0.$$

According to our adopted sign convention (p. 43), we shall have in this case for the infinite beam

$$y_A = y_A' + y_A'', \qquad \theta_A = \theta_A' + \theta_A'',$$

$$y_B = y_A' - y_A'', \qquad \theta_B = -\theta_A' + \theta_A'',$$

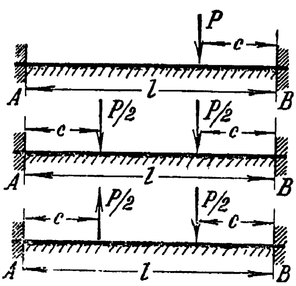

FIG. 39

* In another form:

$$F_I = \frac{1}{2}\,\frac{e^{\lambda l}}{\operatorname{Cosh}\lambda l + \cos\lambda l}, \qquad F_{II} = \frac{1}{2}\,\frac{e^{\lambda l}}{\operatorname{Cosh}\lambda l - \cos\lambda l}.$$

See tables of E and F functions on page 241.

and, therefore,

$$\left.\begin{aligned} y'_A &= \tfrac{1}{2}(y_A + y_B), & \theta'_A &= \tfrac{1}{2}(\theta_A - \theta_B), \\ y''_A &= \tfrac{1}{2}(y_A - y_B), & \theta''_A &= \tfrac{1}{2}(\theta_A + \theta_B). \end{aligned}\right\} \tag{35}$$

Having determined on the infinitely long beam the y_A, y_B, θ_A, and θ_B values due to the given loading, we can obtain the y'_A, θ'_A and y''_A, θ''_A values from (35) for the symmetrical and antisymmetrical components, respectively, of the given loading. They will be used in the formulas derived below.

a. Symmetrical Case

P'_0 and M'_0 must produce $-y'_A$ and $-\theta'_A$ at A (at B, consequently, $-y'_A$ and θ'_A will be produced). This condition requires that

$$\left.\begin{aligned} P'_0\frac{\lambda}{2k}(1 + A_{\lambda l}) + M'_0\frac{\lambda^2}{k}B_{\lambda l} &= -y'_A, \\ P'_0\frac{\lambda^2}{k}B_{\lambda l} + M'_0\frac{\lambda^3}{k}(1 - C_{\lambda l}) &= -\theta'_A. \end{aligned}\right\} \tag{a}$$

From this we have

$$\left.\begin{aligned} P'_0 &= 8\lambda^2 EIE_I[\theta'_A B_{\lambda l} - y'_A\lambda(1 - C_{\lambda l})], \\ M'_0 &= -4\lambda EIE_I[\theta'_A(1 + A_{\lambda l}) - y'_A 2\lambda B_{\lambda l}], \end{aligned}\right\} \tag{36'}$$

where E_I is the same as before (see p. 41).

b. Antisymmetrical Case

P''_0 and M''_0 will produce $-y''_A$ and $-\theta''_A$ at A (thus at B, $+y''_A$ and $-\theta'_A$ will be produced). This condition requires that

$$\left.\begin{aligned} P''_0\frac{\lambda}{2k}(1 - A_{\lambda l}) - M''_0\frac{\lambda^2}{k}B_{\lambda l} &= -y''_A, \\ -P''_0\frac{\lambda^2}{k}B_{\lambda l} + M''_0\frac{\lambda^3}{k}(1 + C_{\lambda l}) &= -\theta''_A \end{aligned}\right\} \tag{b}$$

From this we get

$$\left.\begin{aligned} P''_0 &= -8\lambda^2 EIE_{II}[\theta''_A B_{\lambda l} + \lambda y''_A(1 + C_{\lambda l})], \\ M''_0 &= -4\lambda EIE_{II}[\theta''_A(1 - A_{\lambda l}) + y''_A 2\lambda B_{\lambda l}], \end{aligned}\right\} \tag{36''}$$

where the symbol E_{II} denotes the same expression that was given on page 42.

In an analogous way, the end-conditioning forces can also be determined for other kinds of end conditions than those of free, simply supported, and fixed ends which have been discussed so far.

From the four quantities y, θ, M, and Q we can select two and ascribe any value to them, establishing in this way the conditions for an end. From these four quantities six different types of end support can be derived, a circumstance which, if we consider both ends of the beam, permits of twenty-one different types of beams. Of these, six will be symmetrical, and of those six three have been investigated above. For the remaining three symmetrical beams the analysis can be carried out in the same manner, the general solution being resolved into two pairs of simultaneous equations. But when we deal with different end conditions at the two ends (a situation which admits of fifteen types of beams), such simplification is not possible, and, as a rule, all four equations of the type expressing the four end conditions (see [30]) will have to be solved simultaneously.

17. *Classification of Beams according to Stiffness*

We have seen that it was the λl quantity which characterized the relative stiffness of a beam on an elastic foundation. This λl quantity determines the magnitude of the curvature of the elastic line and defines the rate at which the effect of a loading force dies out in the form of a damped wave along the length of the beam. According to these λl values we may classify beams into three groups:

I. Short beams: $\lambda l < \pi/4$;
II. Beams of medium length: $\pi/4 < \lambda l < \pi$;
III. Long beams: $\lambda l > \pi$.

This classification is made from a practical point of view, since it offers the possibility of using approximations and of neglecting certain quantities in particular instances.

For beams belonging to group I we can neglect, in most practical problems, the bending deformation of the bar, since this deformation will be so small as to be negligible compared with the deformation produced in the foundation. Hence, computing beams of $\lambda l < \pi/4$, we can assume them to be absolutely rigid; consequently, the position which they take on the foundation, when subjected to loading, can be determined from simple considerations of statics.

Group II comprises the situations in which accurate computation of the beams is necessary. The characteristic of this group is that a force acting at one end of the beam has a finite, and not negligible, effect at the other end. Consequently, when a beam of such length is derived from the infinitely long one, the countereffect which the end-conditioning forces have on each other has an important role, and no approximation is advisable.

Beams belonging to group III have a λl value such that the countereffect of the end-conditioning forces on each other is a diminishing one. When investigating one end of the beam, we may assume that the other end is infinitely far away. Forces applied at one end will have a negligible effect at the other. In other words, λl is so large that we can take in all the formulas $A_{\lambda l} = B_{\lambda l} = C_{\lambda l} = D_{\lambda l} = 0$, which greatly simplifies the computation.

The limits established above for the classification of the beams are not definite. They depend on the accuracy required in the computation. Around these limits the suggested approximations will give results differing only by a few per cent from the exact ones. If we want greater accuracy we should put the upper limit of group I at $\lambda l = 0.60$ and the lower limit of group III at $\lambda l = 5.00$. But beyond these limits use should be made of the approximations, since then the difference from the exact results will be negligibly small.

In the example below the exact analysis required for group II is used for a beam which has a value of λl larger than π. This is done in order to show how the accurate solution can be arrived at, but it should be noted that satisfactory final results for practical purposes can be more simply obtained by making use of the approximations.

18. *Example*

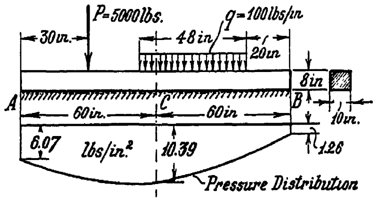

FIG. 40

Consider a beam with free ends, its dimensions and loading as shown in Figure 40. Find the deflections at the ends and the deflection, moment, and shear at C, in the middle. The modulus of elasticity of the material of the beam (wood) is $E = 1.5 \times 10^6$ lbs./in.^{2} and the modulus of the foundation, $k_0 = 200$ lbs./in.^{3} We shall have $k = bk_0 = 10$ in. $\times$ 200 lbs./in.^{3} = 2000 lbs./in.^{2}, $I = 10 \times 8^3/12 = 426.7$ in.^{4},

$$\lambda = \sqrt[4]{\frac{k}{4EI}} \sqrt[4]{\frac{2000 \text{ lbs./in.}^2}{4 \times 1.5 \times 10^6 \text{ lbs./in.}^2 \times 426.7 \text{ in.}^4}} = 0.02973 \text{ in.}^{-1}$$

$\lambda l = 0.02973$ in.$^{-1} \times 120$ in. $= 3.568$.

First we have to calculate the bending moment and shearing force values produced by the given P and q loadings at points A and B on the infinitely long beam.

Force P will produce, according to (5 c–d):

$$\text{at } A\ (x = 30 \text{ in.}):\quad M_A^P = -2548.0 \text{ in. lbs.},\qquad Q_A^P = 643.2 \text{ lbs.};$$

$$\text{at } B\ (x = 90 \text{ in.}):\quad M_B^P = -3889.0 \text{ in. lbs.},\qquad Q_B^P = 153.8 \text{ lbs.}$$

The distributed load q will produce

at A ($a = 52$ in., $b = 100$ in.), according to (8 c–d):

$$M_A^q = -5787.0 \text{ in. lbs.},\qquad Q_A^q = -125.1 \text{ lbs.};$$

at B ($a = 68$ in., $b = 20$ in.), according to (9 c–d):

$$M_B^q = -5402.4 \text{ in. lbs.},\qquad Q_B^q = -272.9 \text{ lbs.}$$

Altogether there will be produced

$$M_A = -8335.0 \text{ in. lbs.},\qquad M_B = -9291.0 \text{ in. lbs.},$$

$$Q_A = 518.1 \text{ lbs.},\qquad Q_B = -119.1 \text{ lbs.}$$

Putting these values into (31), we obtain for the corresponding symmetrical and antisymmetrical components:

$$M'_A = -8813.0 \text{ in. lbs.}, \qquad M''_A = 478.0 \text{ in. lbs.},$$
$$Q'_A = 318.6 \text{ lbs.}, \qquad Q''_A = 199.5 \text{ lbs.}$$

Substituting these values in (32′) and (32″) and taking the $A_{\lambda l}$, $B_{\lambda l}$, etc., values from the tables (p. 235), using interpolation, we obtain the values

$$P'_0 = 158.3 \text{ lbs.}, \qquad M'_0 = 15400.0 \text{ in. lbs.},$$
$$P''_0 = 854.0 \text{ lbs.}, \qquad M''_0 = 15130.0 \text{ in. lbs.}$$

So we shall have the end-conditioning forces

at A: $P_{0A} = P'_0 + P''_0 = 1012.0$ lbs. (downward),

$M_{0A} = M'_0 + M''_0 = 265.0$ in. lbs. (clockwise);

at B: $P_{0B} = P'_0 - P''_0 = 695.7$ lbs. (upward),

$M_{0B} = M'_0 - M''_0 = 30530.0$ in. lbs. (counterclockwise).

These forces and moments applied close to the left of A and to the right of B on the infinitely long beam will produce, according to (5 c–d) and (6 c–d),

at A: $-M_A = 8335.0$ in. lbs., $-Q_A = -518.1$ lbs.;

at B: $-M_B = 9291.0$ in. lbs., $-Q_B = 119.1$ lbs.

Thus, if we apply the end-conditioning forces on the infinite beam simultaneously with the given loading, the deflection, slope, M, and Q will be the same at any point between A and B as in the original problem of Figure 40. This part of the calculation can be carried out conveniently in the following tabular form

a. *Deflections*

	y_A in.	y_C in.	y_B in.
On the infinite beam:			
P	0.02141	0.02141	−0.00113
q	0.00140	0.02797	0.01286
P_{0A}	0.00752	0.00097	−0.00028
M_{0A}	0	0.00002	−0.00000
P_{0B}	0.00019	−0.00066	−0.00517
M_{0B}	−0.00016	0.00222	0
On the beam of finite length:	0.03036	0.05193	0.00628

b. *Pressure in the Foundation*, $p = k_0 y$ lbs./in.^{2}

$p_A = 6.07$ lbs./in.^{2}, $p_C = 10.39$ lbs./in.^{2}, $p_B = 1.26$ lbs./in.^{2}

c. *Bending Moment and Shearing Force at C*

	M_C in. lbs.	Q_C lbs.
On the infinite beam:		
P	−2547.9	−643.2
q	13251.3	630.2
P_{0A}	−1700.0	18.0
M_{0A}	−4.7	−0.5
P_{0B}	1168.2	12.4
M_{0B}	−543.4	58.4
On the beam of finite length:	9623.5	75.3

CHAPTER IV

PARTICULAR CASES OF LOADING ON FINITE BEAMS

I. Solutions of the Differential Equation of the Elastic Line

In this chapter we shall give ready formulas for a number of simple cases of loading on beams of finite length. These expressions were derived by the previously presented method of superposition, by which solutions for any other loading or end conditions can also be obtained. In the formulas, wherever the symbol λ is used, it denotes $\sqrt[4]{k/4EI}$, where $k = bk_0$; k_0 lbs./in.^{3} is the modulus of the foundation; and b is the constant width of the beam in contact with the foundation; and EI is the flexural rigidity of the beam.

19. *Beams with Free Ends*

a. Equal Concentrated Forces at Both Ends (Fig. 41)

$$y = \frac{2P\lambda}{k}\,\frac{\text{Cosh}\,\lambda x \cos\lambda x' + \text{Cosh}\,\lambda x' \cos\lambda x}{\text{Sinh}\,\lambda l + \sin\lambda l}. \tag{37a}$$

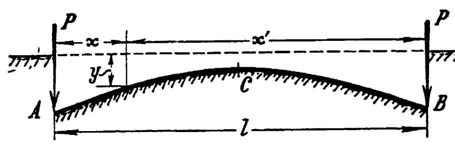

FIG. 41

Deflection at the end points:

$$y_A = y_B = \frac{2P\lambda}{k}\,\frac{\text{Cosh}\,\lambda l + \cos\lambda l}{\text{Sinh}\,\lambda l + \sin\lambda l}.$$

Deflection at the middle:

$$y_C = \frac{4P\lambda}{k}\,\frac{\text{Cosh}\,\frac{\lambda l}{2}\cos\frac{\lambda l}{2}}{\text{Sinh}\,\lambda l + \sin\lambda l}.$$

The deflection $y_C = 0$ when $\lambda l = \pi, 3\pi, 5\pi$, etc.

$$\theta = \frac{2P\lambda^2}{k}\,\frac{1}{\text{Sinh}\,\lambda l + \sin\lambda l}(\text{Sinh}\,\lambda x\cos\lambda x' + \text{Cosh}\,\lambda x \sin\lambda x'$$

$$- \text{Sinh}\,\lambda x' \cos\lambda x - \text{Cosh}\,\lambda x' \sin\lambda x). \tag{37b}$$

Slope at the end points:

$$\theta_A = -\theta_B = -\frac{2P\lambda^2}{k}\,\frac{\text{Sinh}\,\lambda l - \sin\lambda l}{\text{Sinh}\,\lambda l + \sin\lambda l}.$$

$$M = -\frac{P}{\lambda}\,\frac{\text{Sinh}\,\lambda x \sin\lambda x' + \text{Sinh}\,\lambda x' \sin\lambda x}{\text{Sinh}\,\lambda l + \sin\lambda l}. \tag{37c}$$

Bending moment at the middle:

$$M_C = -\frac{2P}{\lambda} \frac{\text{Sinh} \frac{\lambda l}{2} \sin \frac{\lambda l}{2}}{\text{Sinh}\, \lambda l + \sin \lambda l}.$$

$M_C = 0$ when $\lambda l = 2\pi, 4\pi, 6\pi$, etc.
M_C is negative when $\lambda l < 2\pi$.
M_C is positive when $2\pi < \lambda l < 4\pi$.

$$Q = P \frac{1}{\text{Sinh}\, \lambda l + \sin \lambda l} (\text{Sinh}\, \lambda x \cos \lambda x' - \text{Cosh}\, \lambda x \sin \lambda x' + \text{Cosh}\, \lambda x' \sin \lambda x - \text{Sinh}\, \lambda x' \cos \lambda x). \quad (37d)$$

b. Equal Concentrated Moments at Both Ends (Fig. 42)

$$y = -\frac{2M_0\lambda^2}{k} \frac{1}{\text{Sinh}\, \lambda l + \sin \lambda l} (\text{Sinh}\, \lambda x \cos \lambda x' - \text{Cosh}\, \lambda x \sin \lambda x' + \text{Sinh}\, \lambda x' \cos \lambda x - \text{Cosh}\, \lambda x' \sin \lambda x). \quad (38a)$$

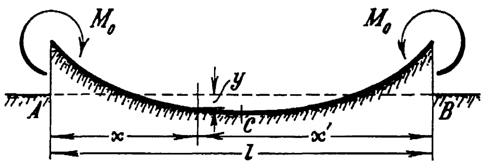

FIG. 42

Deflection at the end points:

$$y_A = y_B = -\frac{2M_0\lambda^2}{k} \frac{\text{Sinh}\, \lambda l - \sin \lambda l}{\text{Sinh}\, \lambda l + \sin \lambda l}.$$

$$\theta = -\frac{4M_0\lambda^3}{k} \frac{\text{Cosh}\, \lambda x \cos \lambda x' - \text{Cosh}\, \lambda x' \cos \lambda x}{\text{Sinh}\, \lambda l + \sin \lambda l}. \quad (38b)$$

Slope at the end points:

$$\theta_A = -\theta_B = \frac{4M_0\lambda^3}{k} \frac{\text{Cosh}\, \lambda l - \cos \lambda l}{\text{Sinh}\, \lambda l + \sin \lambda l}.$$

$$M = M_0 \frac{1}{\text{Sinh}\, \lambda l + \sin \lambda l} (\text{Sinh}\, \lambda x \cos \lambda x' + \text{Cosh}\, \lambda x \sin \lambda x' + \text{Sinh}\, \lambda x' \cos \lambda x + \text{Cosh}\, \lambda x' \sin \lambda x). \quad (38c)$$

Bending moment at the middle:

$$M_C = 2M_0 \frac{\text{Sinh} \frac{\lambda l}{2} \cos \frac{\lambda l}{2} + \text{Cosh} \frac{\lambda l}{2} \sin \frac{\lambda l}{2}}{\text{Sinh}\, \lambda l + \sin \lambda l}.$$

$M_C = 0$ when $\cos\frac{\lambda l}{2} = -\sin\frac{\lambda l}{2}$, that is, when $\lambda l = \frac{3}{2}\pi, \frac{7}{2}\pi, \frac{11}{2}\pi$, etc., if we assume that at these λl values $\text{Sinh}\,\lambda l \approx \text{Cosh}\,\lambda l$.

$$Q = 2M_0\lambda\frac{\text{Sinh}\,\lambda x\sin\lambda x' - \text{Sinh}\,\lambda x'\sin\lambda x}{\text{Sinh}\,\lambda l + \sin\lambda l}. \tag{38d}$$

c. Concentrated Force at One End (Fig. 43)

$$y = \frac{2P\lambda}{k}\frac{\text{Sinh}\,\lambda l\cos\lambda x\,\text{Cosh}\,\lambda x' - \sin\lambda l\,\text{Cosh}\,\lambda x\cos\lambda x'}{\text{Sinh}^2\lambda l - \sin^2\lambda l}. \tag{39a}$$

Deflections at the end points:

$$y_A = \frac{2P\lambda}{k}\frac{\text{Sinh}\,\lambda l\,\text{Cosh}\,\lambda l - \sin\lambda l\cos\lambda l}{\text{Sinh}^2\lambda l - \sin^2\lambda l},$$

$$y_B = \frac{2P\lambda}{k}\frac{\text{Sinh}\,\lambda l\cos\lambda l - \sin\lambda l\,\text{Cosh}\,\lambda l}{\text{Sinh}^2\lambda l - \sin^2\lambda l}.$$

The deflection $y_B = 0$ when $\lambda l = \frac{5}{4}\pi, \frac{9}{4}\pi$, etc., if we assume that at these λl values $\text{Sinh}\,\lambda l \approx \text{Cosh}\,\lambda l$.

$$\theta = -\frac{2P\lambda^2}{k}\frac{1}{\text{Sinh}^2\lambda l - \sin^2\lambda l}[\text{Sinh}\,\lambda l\,(\sin\lambda x\,\text{Cosh}\,\lambda x' + \cos\lambda x\,\text{Sinh}\,\lambda x')$$

$$+ \sin\lambda l\,(\text{Sinh}\,\lambda x\cos\lambda x' + \text{Cosh}\,\lambda x\sin\lambda x')]. \tag{39b}$$

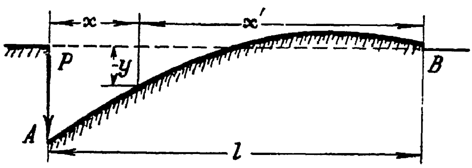

FIG. 43

Slopes at the end points:

$$\theta_A = -\frac{2P\lambda^2}{k}\frac{\text{Sinh}^2\lambda l + \sin^2\lambda l}{\text{Sinh}^2\lambda l - \sin^2\lambda l},$$

$$\theta_B = -\frac{4P\lambda^2}{k}\frac{\text{Sinh}\,\lambda l\sin\lambda l}{\text{Sinh}^2\lambda l - \sin^2\lambda l}.$$

The slope $\theta_B = 0$ when $\lambda l = \pi, 2\pi, 3\pi$, etc.

$$M = -\frac{P}{\lambda}\frac{\text{Sinh}\,\lambda l\sin\lambda x\,\text{Sinh}\,\lambda x' - \sin\lambda l\,\text{Sinh}\,\lambda x\sin\lambda x'}{\text{Sinh}^2\lambda l - \sin^2\lambda l}. \tag{39c}$$

$$Q = -P\frac{1}{\text{Sinh}^2\lambda l - \sin^2\lambda l}[\text{Sinh}\,\lambda l\,(\cos\lambda x\,\text{Sinh}\,\lambda x' - \sin\lambda x\,\text{Cosh}\,\lambda x')$$

$$- \sin\lambda l\,(\text{Cosh}\,\lambda x\sin\lambda x' - \text{Sinh}\,\lambda x\cos\lambda x')]. \tag{39d}$$

d. Concentrated Moment at One End (Fig. 44)

$$y = \frac{2M_0\lambda^2}{k}\frac{1}{\text{Sinh}^2\lambda l - \sin^2\lambda l}[\text{Sinh}\,\lambda l\,(\text{Cosh}\,\lambda x'\sin\lambda x - \text{Sinh}\,\lambda x'\cos\lambda x)$$

$$+ \sin\lambda l\,(\text{Sinh}\,\lambda x\cos\lambda x' - \text{Cosh}\,\lambda x\sin\lambda x')]. \tag{40a}$$

Deflections at the end points:

$$y_A = -\frac{2M_0\lambda^2}{k} \frac{\mathrm{Sinh}^2 \lambda l + \sin^2 \lambda l}{\mathrm{Sinh}^2 \lambda l - \sin^2 \lambda x},$$

$$y_B = \frac{4M_0\lambda^2}{k} \frac{\mathrm{Sinh}\, \lambda l \sin \lambda l}{\mathrm{Sinh}^2 \lambda l - \sin^2 \lambda l}.$$

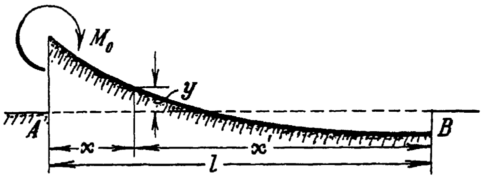

FIG. 44

The deflection $y_B = 0$ when $\lambda l = \pi, 2\pi, 3\pi$, etc.

$$\theta = \frac{4M_0\lambda^3}{k} \frac{\mathrm{Sinh}\, \lambda l \,\mathrm{Cosh}\, \lambda x' \cos \lambda x + \sin \lambda l \,\mathrm{Cosh}\, \lambda x \cos \lambda x'}{\mathrm{Sinh}^2 \lambda l - \sin^2 \lambda l}. \tag{40b}$$

Slopes at the end points:

$$\theta_A = \frac{4M_0\lambda^3}{k} \frac{\mathrm{Sinh}\, \lambda l \,\mathrm{Cosh}\, \lambda l + \sin \lambda l \cos \lambda l}{\mathrm{Sinh}^2 \lambda l - \sin^2 \lambda l}$$

$$\theta_B = \frac{4M_0\lambda^3}{k} \frac{\mathrm{Sinh}\, \lambda l \cos \lambda l + \sin \lambda l \,\mathrm{Cosh}\, \lambda l}{\mathrm{Sinh}^2 \lambda l - \sin^2 \lambda l}.$$

The slope $\theta_B = 0$ when $\lambda l = \frac{3}{4}\pi, \frac{7}{4}\pi, \frac{11}{4}\pi$, etc., if we assume that at these λl values $\mathrm{Sinh}\, \lambda l \approx \mathrm{Cosh}\, \lambda l$.

$$M = M_0 \frac{1}{\mathrm{Sinh}^2 \lambda l - \sin^2 \lambda l} [\mathrm{Sinh}\, \lambda l\, (\mathrm{Sinh}\, \lambda x' \cos \lambda x + \mathrm{Cosh}\, \lambda x' \sin \lambda x)$$
$$- \sin \lambda l\, (\mathrm{Sinh}\, \lambda x \cos \lambda x' + \mathrm{Cosh}\, \lambda x \sin \lambda x')]. \tag{40c}$$

$$Q = -2M_0\lambda \frac{\mathrm{Sinh}\, \lambda l \,\mathrm{Sinh}\, \lambda x' \sin \lambda x + \sin \lambda l \,\mathrm{Sinh}\, \lambda x \sin \lambda x'}{\mathrm{Sinh}^2 \lambda l - \sin^2 \lambda l}. \tag{40d}$$

e. Concentrated Force at the Middle (Figs. 45–46)

$$y = \frac{P\lambda}{2k} \frac{1}{\mathrm{Sinh}\, \lambda l + \sin \lambda l} [\mathrm{Cosh}\, \lambda x \cos \lambda(l - x) + \cos \lambda x \,\mathrm{Cosh}\, \lambda(l - x)$$
$$- \mathrm{Sinh}\, \lambda x \sin \lambda(l - x) + \sin \lambda x \,\mathrm{Sinh}\, \lambda(l - x)$$
$$+ 2 \,\mathrm{Cosh}\, \lambda x \cos \lambda x]. \tag{41a}$$

Deflection at the middle:

$$y_C = \frac{P\lambda}{2k} \frac{\mathrm{Cosh}\, \lambda l + \cos \lambda l + 2}{\mathrm{Sinh}\, \lambda l + \sin \lambda l}.$$

Deflection at the end points:

$$y_A = y_B = \frac{2P\lambda}{k} \frac{\mathrm{Cosh}\, \frac{\lambda l}{2} \cos \frac{\lambda l}{2}}{\mathrm{Sinh}\, \lambda l + \sin \lambda l}.$$

The deflection $y_A = y_B = 0$ when $\lambda l = \pi, 3\pi, 5\pi$, etc.

The deflection at the end points is positive when $\lambda l < \pi$; $\lambda l = \pi$ defines the *effective length* of the beam subjected to a concentrated force in the middle (Fig. 46).

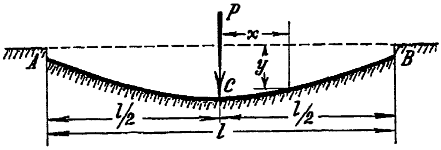

FIG. 45

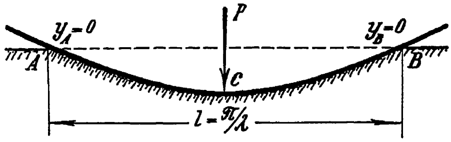

FIG. 46

$$\theta = \frac{P\lambda^2}{k}\frac{1}{\text{Sinh}\,\lambda l + \sin\lambda l}\{\text{Sinh}\,\lambda x\,[\cos\lambda x + \cos\lambda(l - x)] - \sin\lambda x\,[\text{Cosh}\,\lambda x + \text{Cosh}\,\lambda(l - x)]\}. \tag{41b}$$

Slope at the end points

$$\theta_A = -\theta_B = -\frac{2P\lambda^2}{k}\frac{\text{Sinh}\,\frac{\lambda l}{2}\cos\frac{\lambda l}{2} - \sin\frac{\lambda l}{2}\,\text{Cosh}\,\frac{\lambda l}{2}}{\text{Sinh}\,\lambda l + \sin\lambda l}.$$

$$M = \frac{P}{4\lambda}\frac{1}{\text{Sinh}\,\lambda l + \sin\lambda l}\{\text{Sinh}\,\lambda x\,[\sin\lambda x - \sin\lambda(l - x)] - \text{Cosh}\,\lambda x\,[\cos\lambda x + \cos\lambda(l - x)] + \sin\lambda x\,[\text{Sinh}\,\lambda x - \text{Sinh}\,\lambda(l - x)] + \cos\lambda x\,[\text{Cosh}\,\lambda x + \text{Cosh}\,\lambda(l - x)]\}. \tag{41c}$$

Maximum bending moment at the middle:

$$M_C = \frac{P}{4\lambda}\frac{\text{Cosh}\,\lambda l - \cos\lambda l}{\text{Sinh}\,\lambda l + \sin\lambda l}.$$

$$Q = \frac{P}{2}\frac{1}{\text{Sinh}\,\lambda l + \sin\lambda l}\{\text{Cosh}\,\lambda x\,[\sin\lambda x - \sin\lambda(l - x)] + \cos\lambda x\,[\text{Sinh}\,\lambda x - \text{Sinh}\,\lambda(l - x)]\}. \tag{41d}$$

f. Concentrated Force at an Arbitrary Point (Fig. 47)

The following formulas for the deflection, slope, bending-moment, and shearing-force curves are for the A–C portion of the beam, where $x < a$. The same formulas can be used for the B–C section, where $x < b$, by measuring x from end B and replacing a by b and b by a,

$$y = \frac{P\lambda}{k}\frac{1}{\text{Sinh}^2\,\lambda l - \sin^2\lambda l}\{2\,\text{Cosh}\,\lambda x\cos\lambda x(\text{Sinh}\,\lambda l\cos\lambda a\,\text{Cosh}\,\lambda b - \sin\lambda l\,\text{Cosh}\,\lambda a\cos\lambda b) + (\text{Cosh}\,\lambda x\sin\lambda x + \text{Sinh}\,\lambda x\cos\lambda x)[\text{Sinh}\,\lambda l(\sin\lambda a\,\text{Cosh}\,\lambda b - \cos\lambda a\,\text{Sinh}\,\lambda b) + \sin\lambda l(\text{Sinh}\,\lambda a\cos\lambda b - \text{Cosh}\,\lambda a\sin\lambda b)]\}. \tag{42a}$$

Deflection at the point of application of the load ($x = a$):

$$y_C = \frac{P\lambda}{k}\frac{1}{\text{Sinh}^2 \lambda l - \sin^2 \lambda l}[(\text{Cosh}^2 \lambda a + \cos^2 \lambda a)(\text{Sinh } \lambda b \text{ Cosh } \lambda b - \sin \lambda b \cos \lambda b) + (\text{Cosh}^2 \lambda b + \cos^2 \lambda b)(\text{Sinh } \lambda a \text{ Cosh } \lambda a - \sin \lambda a \cos \lambda a)].$$

$$\theta = \frac{2P\lambda^2}{k}\frac{1}{\text{Sinh}^2 \lambda l - \sin^2 \lambda l}\{\text{Cosh } \lambda x \cos \lambda x \cdot [\text{Sinh } \lambda l(\sin \lambda a \text{ Cosh } \lambda b - \cos \lambda a \text{ Sinh } \lambda b) + \sin \lambda l(\text{Sinh } \lambda a \cos \lambda b - \text{Cosh } \lambda a \sin \lambda b)] - (\text{Cosh } \lambda x \sin \lambda x - \text{Sinh } \lambda x \cos \lambda x) \cdot (\text{Sinh } \lambda l \cos \lambda a \text{ Cosh } \lambda b - \sin \lambda l \text{ Cosh } \lambda a \cos \lambda b)\}. \quad (42b)$$

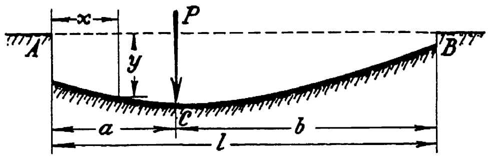

FIG. 47

Slope at the point of application of the load ($x = a$):

$$\theta_C = \frac{2P\lambda^2}{k}\frac{\text{Cosh}^2 \lambda a \cos^2 \lambda b - \cos^2 \lambda a \text{ Cosh}^2 \lambda b}{\text{Sinh}^2 \lambda l - \sin^2 \lambda l}.$$

$$M = \frac{P}{2\lambda}\frac{1}{\text{Sinh}^2 \lambda l - \sin^2 \lambda l}\{2 \text{ Sinh } \lambda x \sin \lambda x(\text{Sinh } \lambda l \cos \lambda a \text{ Cosh } \lambda b - \sin \lambda l \text{ Cosh } \lambda a \cos \lambda b) + (\text{Cosh } \lambda x \sin \lambda x - \text{Sinh } \lambda x \cos \lambda x)[\text{Sinh } \lambda l(\sin \lambda a \text{ Cosh } \lambda b - \cos \lambda a \text{ Sinh } \lambda b) + \sin \lambda l(\text{Sinh } \lambda a \cos \lambda b - \text{Cosh } \lambda a \sin \lambda b)]\}. \quad (42c)$$

Bending moment at the point of application of the load ($x = a$):

$$M_C = \frac{P}{4\lambda}\frac{1}{\text{Sinh}^2 \lambda l - \sin^2 \lambda l}[(\text{Cosh}^2 \lambda a - \cos^2 \lambda a)(\text{Sinh } 2\lambda b - \sin 2\lambda b) + (\text{Cosh}^2 \lambda b - \cos^2 \lambda b)(\text{Sinh } 2\lambda a - \sin 2\lambda a)]$$

$$Q = P\frac{1}{\text{Sinh}^2 \lambda l - \sin^2 \lambda l}\{(\text{Cosh } \lambda x \sin \lambda x + \text{Sinh } \lambda x \cos \lambda x) \cdot (\text{Sinh } \lambda l \cos \lambda a \text{ Cosh } \lambda b - \sin \lambda l \text{ Cosh } \lambda a \cos \lambda b) + \text{Sinh } \lambda x \sin \lambda x[\text{Sinh } \lambda l(\sin \lambda a \text{ Cosh } \lambda b - \cos \lambda a \text{ Sinh } \lambda b) + \sin \lambda l(\text{Sinh } \lambda a \cos \lambda b - \text{Cosh } \lambda a \sin \lambda b)]\}. \quad (42d)$$

Shearing force directly to the left of point C:

$$Q_C = \frac{P}{4}\frac{1}{\operatorname{Sinh}^2 \lambda l - \sin^2 \lambda l}(4 \operatorname{Sinh} \lambda l \operatorname{Sinh} \lambda a \operatorname{Cosh} \lambda b - 4 \sin \lambda l \sin \lambda a \cos \lambda b - \operatorname{Sinh} 2\lambda a \sin 2\lambda b + \sin 2\lambda a \operatorname{Sinh} 2\lambda b).$$

g. Two Equal Concentrated Forces Symmetrically Placed (Fig. 48)

Deflection line for portion A–C ($x < a$):

$$y_{A-C} = \frac{P\lambda}{k}\frac{1}{\operatorname{Sinh} \lambda l + \sin \lambda l}\{2 \operatorname{Cosh} \lambda x \cos \lambda x[\operatorname{Cosh} \lambda a \cos \lambda(l - a) + \operatorname{Cosh} \lambda(l - a) \cos \lambda a] + (\operatorname{Cosh} \lambda x \sin \lambda x + \operatorname{Sinh} \lambda x \cos \lambda x) \cdot [\operatorname{Cosh} \lambda a \sin \lambda(l - a) - \operatorname{Sinh} \lambda a \cos \lambda(l - a) + \operatorname{Cosh} \lambda(l - a) \sin \lambda a - \operatorname{Sinh} \lambda(l - a) \cos \lambda a]\}. \quad (43a')$$

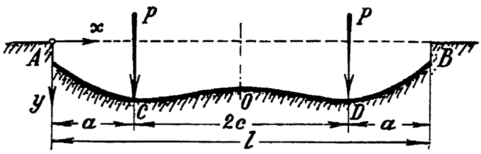

FIG. 48

With the aid of the formula above the deflection line for the portion C–D ($a < x < l - a$) can be expressed as

$$y_{C-D} = [y_{A-C}]_{x>a} + \frac{P\lambda}{k}[\operatorname{Cosh} \lambda(x - a) \sin \lambda(x - a) - \operatorname{Sinh} \lambda(x - a) \cos \lambda(x - a)]. \quad (43a'')$$

Particular values of deflection ordinates:

$$y_A = y_B = \frac{2P\lambda}{k}\frac{\operatorname{Cosh} \lambda a \cos \lambda(l - a) + \operatorname{Cosh} \lambda(l - a) \cos \lambda a}{\operatorname{Sinh} \lambda l + \sin \lambda l},$$

$$y_C = y_D = \frac{P\lambda}{2k}\frac{1}{\operatorname{Sinh} \lambda l + \sin \lambda l}[2 \operatorname{Cosh}^2 \lambda a(\cos 2\lambda c + \operatorname{Cosh} \lambda l) + 2 \cos^2 \lambda a(\operatorname{Cosh} 2\lambda c + \cos \lambda l) + \operatorname{Sinh} 2\lambda a(\sin 2\lambda c - \operatorname{Sinh} \lambda l) - \sin 2\lambda a(\operatorname{Sinh} 2\lambda c - \sin \lambda l)],$$

$$y_O = \frac{P\lambda}{k}\frac{1}{\operatorname{Sinh} \lambda l + \sin \lambda l}\{\operatorname{Cosh} \lambda c[\cos \lambda(l - c) + \cos \lambda c] + \cos \lambda c[\operatorname{Cosh} \lambda(l - c) + \operatorname{Cosh} \lambda c] - \operatorname{Sinh} \lambda c \sin \lambda(l - c) + \sin \lambda c \operatorname{Sinh} \lambda(l - c)\}.$$

Angular deflection along A–C ($x < a$):

$$\theta_{A-C} = \frac{2P\lambda^2}{k}\frac{1}{\text{Sinh}\,\lambda l + \sin\lambda l}\{(\text{Sinh}\,\lambda x\cos\lambda x - \text{Cosh}\,\lambda x\sin\lambda x)$$
$$\cdot[\text{Cosh}\,\lambda a\cos\lambda(l-a) + \text{Cosh}\,\lambda(l-a)\cos\lambda a]$$
$$+ \text{Cosh}\,\lambda x\cos\lambda x[\text{Cosh}\,\lambda a\sin\lambda(l-a) - \text{Sinh}\,\lambda a\cos\lambda(l-a)$$
$$+ \text{Cosh}\,\lambda(l-a)\sin\lambda a - \text{Sinh}\,\lambda(l-a)\cos\lambda a]\}. \quad (43b')$$

For portion C–D ($a < x < l - a$) we have

$$\theta_{C-D} = [\theta_{A-C}]_{x>a} - \frac{2P\lambda^2}{k}\text{Sinh}\,\lambda(x-a)\sin\lambda(x-a). \quad (43b'')$$

Particular values of θ:

$$\theta_A = -\theta_B = \frac{2P\lambda^2}{k}\frac{1}{\text{Sinh}\,\lambda l + \sin\lambda l}[\text{Cosh}\,\lambda a\sin\lambda(l-a)$$
$$- \text{Sinh}\,\lambda a\cos\lambda(l-a) + \text{Cosh}\,\lambda(l-a)\sin\lambda a - \text{Sinh}\,\lambda(l-a)\cos\lambda a]$$

$$\theta_C = -\theta_D = \frac{2P\lambda^2}{k}\frac{\text{Cosh}^2\lambda a\sin 2\lambda c - \cos^2\lambda a\,\text{Sinh}\,2\lambda c}{\text{Sinh}\,\lambda l + \sin\lambda l}.$$

The condition for $\theta_C = 0$ (a matter very important in railroad ties) will be

$$\frac{\text{Cosh}\,\lambda a}{\cos\lambda a} = \sqrt{\frac{\text{Sinh}\,2\lambda c}{\sin 2\lambda c}}.$$

This condition is approximately satisfied when $a = 0.82\,c$.

Bending moment along portion A–C ($x < a$):

$$M_{A-C} = \frac{P}{2\lambda}\frac{1}{\text{Sinh}\,\lambda l + \sin\lambda l}\{2\,\text{Sinh}\,\lambda x\sin\lambda x[\text{Cosh}\,\lambda a\cos\lambda(l-a)$$
$$+ \text{Cosh}\,\lambda(l-a)\cos\lambda a] + (\text{Cosh}\,\lambda x\sin\lambda x$$
$$- \text{Sinh}\,\lambda x\cos\lambda x)[\text{Cosh}\,\lambda a\sin\lambda(l-a) - \text{Sinh}\,\lambda a\cos\lambda(l-a)$$
$$+ \text{Cosh}\,\lambda(l-a)\sin\lambda a - \text{Sinh}\,\lambda(l-a)\cos\lambda a]\}. \quad (43c')$$

For portion C–D ($a < x < l - a$) we have

$$M_{C-D} = [M_{A-C}]_{x>a} - \frac{P}{2\lambda}[\text{Cosh}\,\lambda(x-a)\sin\lambda(x-a)$$
$$+ \text{Sinh}\,\lambda(x-a)\cos\lambda(x-a)]. \quad (43c'')$$

Particular values of M:

$$M_C = M_D = \frac{P}{4\lambda}\frac{1}{\text{Sinh}\,\lambda l + \sin\lambda l}[2\,\text{Cosh}^2\lambda a(\cos 2\lambda c + \text{Cosh}\,\lambda l)$$
$$- 2\cos^2\lambda a(\text{Cosh}\,2\lambda c + \cos\lambda l)$$
$$- \text{Sinh}\,2\lambda a(\sin 2\lambda c + \text{Sinh}\,\lambda l)$$
$$- \sin 2\lambda a(\text{Sinh}\,2\lambda c + \sin\lambda l)],$$

$$M_0 = -\frac{P}{2\lambda}\frac{1}{\text{Sinh}\,\lambda l + \sin\lambda l}\{\text{Sinh}\,\lambda c[\sin\lambda c + \sin\lambda(l-c)] + \sin\lambda c[\text{Sinh}\,\lambda c + \text{Sinh}\,\lambda(l-c)] + \text{Cosh}\,\lambda c\cos\lambda(l-c) - \cos\lambda c\,\text{Cosh}\,\lambda(l-c)\}.$$

Shearing force along A–C ($x < a$):

$$Q_{A-C} = P\frac{1}{\text{Sinh}\,\lambda l + \sin\lambda l}\{(\text{Sinh}\,\lambda x\cos\lambda x + \text{Cosh}\,\lambda x\sin\lambda x) \cdot[\text{Cosh}\,\lambda a\cos\lambda(l-a) + \text{Cosh}\,\lambda(l-a)\cos\lambda a] + \text{Sinh}\,\lambda x\sin\lambda x[\text{Cosh}\,\lambda a\sin\lambda(l-a) - \text{Sinh}\,\lambda a\cos\lambda(l-a) + \text{Cosh}\,\lambda(l-a)\sin\lambda a - \text{Sinh}\,\lambda(l-a)\cos\lambda a]\}. \quad (43d')$$

For portion C–D ($a < x < l - a$) we have

$$Q_{C-D} = [Q_{A-C}]_{x>a} + P\,\text{Cosh}\,\lambda(x-a)\cos\lambda(x-a). \quad (43d'')$$

Shearing force directly to the left of point C:

$$Q_C = \frac{P}{2}\frac{1}{\text{Sinh}\,\lambda l + \sin\lambda l}[\text{Sinh}\,2\lambda a(\cos 2\lambda c + \text{Cosh}\,\lambda l) + \sin 2\lambda a(\text{Cosh}\,2\lambda c + \cos\lambda l) - 2\,\text{Sinh}^2\lambda a\,\text{Sinh}\,\lambda l + 2\sin^2\lambda a\sin\lambda l].$$

h. Symmetrically Placed Uniformly Distributed Loading (Fig. 49)

Deflection line for portion A–C ($x < a$):

$$y_{A-C} = \frac{q}{k}\frac{1}{\text{Sinh}\,\lambda l + \sin\lambda l}\{\text{Cosh}\,\lambda x\cos\lambda x[\text{Cosh}\,\lambda a\sin\lambda(l-a) - \text{Sinh}\,\lambda a\cos\lambda(l-a) + \cos\lambda a\,\text{Sinh}\,\lambda(l-a) - \sin\lambda a\,\text{Cosh}\,\lambda(l-a)] + (\text{Cosh}\,\lambda x\sin\lambda x + \text{Sinh}\,\lambda x\cos\lambda x)[\sin\lambda a\,\text{Sinh}\,\lambda(l-a) - \text{Sinh}\,\lambda a\sin\lambda(l-a)]\}. \quad (44a')$$

With the aid of the formula above the deflection line for the portion C–D can be expressed as

$$y_{C-D} = [y_{A-C}]_{x>a} + \frac{q}{k}[1 - \text{Cosh}\,\lambda(x-a)\cos\lambda(x-a)]. \quad (44a'')$$

Deflection at the end points:

$$y_A = y_B = \frac{q}{k}\frac{1}{\text{Sinh}\,\lambda l + \sin\lambda l}[\text{Cosh}\,\lambda a\sin\lambda(l-a) - \text{Sinh}\,\lambda a\cos\lambda(l-a) + \cos\lambda a\,\text{Sinh}\,\lambda(l-a) - \sin\lambda a\,\text{Cosh}\,\lambda(l-a)].$$

Deflection at the middle:

$$y_O = \frac{q}{k}\left[1 - \frac{2\left(\text{Sinh}\ \lambda a \cos \lambda c\ \text{Cosh}\ \frac{\lambda l}{2} + \sin \lambda a\ \text{Cosh}\ \lambda c \cos \frac{\lambda l}{2}\right)}{\text{Sinh}\ \lambda l + \sin \lambda l}\right]$$

Expressions for slope, bending moment, and shearing force can be derived by differentiating repeatedly with respect to x the formula ([44a′] or [44a″]) for the deflection line. This yields the particular values below.

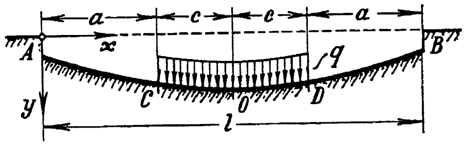

FIG. 49

Slope at the ends:

$$\theta_A = -\theta_B = \frac{2q\lambda}{k}\,\frac{\sin \lambda a\ \text{Sinh}\ \lambda(l-a) - \text{Sinh}\ \lambda a \sin \lambda(l-a)}{\text{Sinh}\ \lambda l + \sin \lambda l}.$$

Bending moment at the middle:

$$M_O = \frac{q}{\lambda^2}\,\frac{\sin \lambda c\ \text{Sinh}\ \frac{\lambda l}{2}\ \text{Sinh}\ \lambda a + \text{Sinh}\ \lambda c \sin \frac{\lambda l}{2} \sin \lambda a}{\text{Sinh}\ \lambda l + \sin \lambda l}.$$

20. *Beams with Hinged Ends*

a. Concentrated Force at the Middle (Fig. 50)

$$y = \frac{P\lambda}{2k(\text{Cosh}\ \lambda l + \cos \lambda l)}[\cos \lambda x\ \text{Sinh}\ \lambda(l-x) - \text{Cosh}\ \lambda x \sin \lambda(l-x) + \sin \lambda x\ \text{Cosh}\ \lambda(l-x) - \text{Sinh}\ \lambda x \cos \lambda(l-x)]. \quad (45a)$$

Deflection at the middle:

$$y_C = \frac{P\lambda}{2k}\,\frac{\text{Sinh}\ \lambda l - \sin \lambda l}{\text{Cosh}\ \lambda l + \cos \lambda l}.$$

$$\theta = -\frac{P\lambda^2}{k}\,\frac{1}{\text{Cosh}\ \lambda l + \cos \lambda l}[\text{Sinh}\ \lambda x \sin \lambda(l-x) + \sin \lambda x\ \text{Sinh}\ \lambda(l-x)]. \quad (45b)$$

Slope at the end points:

$$\theta_A = -\theta_B = \frac{2P\lambda^2}{k}\,\frac{\text{Sinh}\ \frac{\lambda l}{2} \sin \frac{\lambda l}{2}}{\text{Cosh}\ \lambda l + \cos \lambda l}.$$

$$M = \frac{P}{4\lambda}\,\frac{1}{\text{Cosh}\ \lambda l + \cos \lambda l}[\text{Cosh}\ \lambda x \sin \lambda(l-x) - \text{Sinh}\ \lambda x \cos \lambda(l-x) + \cos \lambda x\ \text{Sinh}\ \lambda(l-x) - \sin \lambda x\ \text{Cosh}\ \lambda(l-x)]. \quad (45c)$$

Bending moment at the middle:

$$M_C = \frac{P}{4\lambda} \frac{\text{Sinh } \lambda l + \sin \lambda l}{\text{Cosh } \lambda l + \cos \lambda l}.$$

$$Q = -\frac{P}{2} \frac{1}{\text{Cosh } \lambda l + \cos \lambda l} [\text{Cosh } \lambda x \cos \lambda(l - x) + \cos \lambda x \text{ Cosh } \lambda(l - x)]. \quad (45d)$$

From this we can obtain an expression for the reactions $A = B = -[Q]_{x=l/2}$; if positive values thus denote upward-pointing reaction forces, we have

$$A = B = P \frac{\text{Cosh } \frac{\lambda l}{2} \cos \frac{\lambda l}{2}}{\text{Cosh } \lambda l + \cos \lambda l}.$$

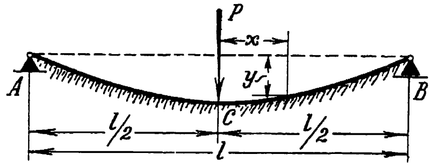

FIG. 50

$A = B = 0$ when $\lambda l = \pi, 3\pi, 5\pi$, etc.

The first root $\lambda l = \pi$ corresponds to the so-called effective length discussed on page 54.

b. Uniformly Distributed Loading over the Whole Span (Fig. 51)

$$y = \frac{q}{k}\left(1 - \frac{\text{Cosh } \lambda x \cos \lambda x' + \text{Cosh } \lambda x' \cos \lambda x}{\text{Cosh } \lambda l + \cos \lambda l}\right). \quad (46a)$$

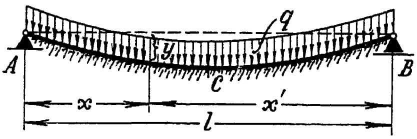

FIG. 51

Deflection at the middle:

$$y_C = \frac{q}{k}\left(1 - \frac{2 \text{ Cosh } \frac{\lambda l}{2} \cos \frac{\lambda l}{2}}{\text{Cosh } \lambda l + \cos \lambda l}\right).$$

$$\theta = \frac{q\lambda}{k} \frac{1}{\text{Cosh } \lambda l + \cos \lambda l} (\text{Sinh } \lambda x \cos \lambda x' + \text{Cosh } \lambda x \sin \lambda x'$$
$$- \text{Sinh } \lambda x' \cos \lambda x - \text{Cosh } \lambda x' \sin \lambda x). \quad (46b)$$

Slope at the end points:

$$\theta_A = -\theta_B = \frac{q\lambda}{k} \frac{\text{Sinh } \lambda l - \sin \lambda l}{\text{Cosh } \lambda l + \cos \lambda l}.$$

$$M = \frac{q}{2\lambda^2} \frac{\text{Sinh } \lambda x \sin \lambda x' + \text{Sinh } \lambda x' \sin \lambda x}{\text{Cosh } \lambda l + \cos \lambda l}. \quad (46c)$$

Bending moment at the middle:

$$M_C = \frac{q}{\lambda^2} \frac{\text{Sinh } \frac{\lambda l}{2} \sin \frac{\lambda l}{2}}{\text{Cosh } \lambda l + \cos \lambda l}.$$

$$Q = -\frac{q}{2\lambda} \frac{1}{\text{Cosh } \lambda l + \cos \lambda l} (\text{Sinh } \lambda x \cos \lambda x' - \text{Cosh } \lambda x \sin \lambda x'$$
$$+ \text{Cosh } \lambda x' \sin \lambda x - \text{Sinh } \lambda x' \cos \lambda x). \quad (46d)$$

Hence we obtain an expression for the reactions $A = B = [Q]_{\substack{x=0 \\ x'=l}}$;

$$A = B = \frac{q}{2\lambda} \frac{\text{Sinh}\,\lambda l + \sin \lambda l}{\text{Cosh}\,\lambda l + \cos \lambda l}.$$

c. Equal Concentrated Moments at Both Ends (Fig. 52)

$$y = \frac{2M_0\lambda^2}{k} \frac{\text{Sinh}\,\lambda x' \sin \lambda x + \text{Sinh}\,\lambda x \sin \lambda x'}{\text{Cosh}\,\lambda l + \cos \lambda l}. \tag{47a}$$

Deflection at the middle:

$$y_C = \frac{4M_0\lambda^2}{k} \frac{\text{Sinh}\,\frac{\lambda l}{2} \sin \frac{\lambda l}{2}}{\text{Cosh}\,\lambda l + \cos \lambda l}.$$

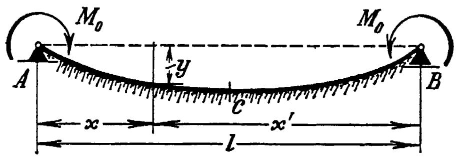

FIG. 52

$$\theta = \frac{2M_0\lambda^3}{k} \frac{1}{\text{Cosh}\,\lambda l + \cos \lambda l} (\text{Cosh}\,\lambda x \sin \lambda x' - \text{Sinh}\,\lambda x \cos \lambda x'$$
$$- \text{Cosh}\,\lambda x' \sin \lambda x + \text{Sinh}\,\lambda x' \cos \lambda x). \tag{47b}$$

Slope at the end points:

$$\theta_A = -\theta_B = \frac{2M_0\lambda^3}{k} \frac{\text{Sinh}\,\lambda l + \sin \lambda l}{\text{Cosh}\,\lambda l + \cos \lambda l}.$$

$$M = M_0 \frac{\text{Cosh}\,\lambda x \cos \lambda x' + \text{Cosh}\,\lambda x' \cos \lambda x}{\text{Cosh}\,\lambda l + \cos \lambda l}. \tag{47c}$$

Bending moment at the middle:

$$M_C = 2M_0 \frac{\text{Cosh}\,\frac{\lambda l}{2} \cos \frac{\lambda l}{2}}{\text{Cosh}\,\lambda l + \cos \lambda l}.$$

$$Q = M_0\lambda \frac{1}{\text{Cosh}\,\lambda l + \cos \lambda l} (\text{Sinh}\,\lambda x \cos \lambda x' + \text{Cosh}\,\lambda x \sin \lambda x'$$
$$- \sin \lambda x\, \text{Cosh}\,\lambda x' - \cos \lambda x\, \text{Sinh}\,\lambda x'). \tag{47d}$$

Reaction forces:

$$A = B = -M_0\lambda \frac{\text{Sinh}\,\lambda l - \sin \lambda l}{\text{Cosh}\,\lambda l + \cos \lambda l}.$$

d. Concentrated Moment at One End (Fig. 53)

$$y = \frac{2M_0\lambda^2}{k} \frac{\text{Cosh}\,\lambda l\, \text{Sinh}\,\lambda x' \sin \lambda x - \cos \lambda l\, \text{Sinh}\,\lambda x \sin \lambda x'}{\text{Cosh}^2 \lambda l - \cos^2 \lambda l}. \tag{48a}$$

$$\theta = \frac{2M_0\lambda^3}{k} \frac{1}{\text{Cosh}^2 \lambda l - \cos^2 \lambda l} [\text{Cosh}\,\lambda l(\cos \lambda x\, \text{Sinh}\,\lambda x' - \sin \lambda x\, \text{Cosh}\,\lambda x')$$
$$- \cos \lambda l(\text{Cosh}\,\lambda x \sin \lambda x' - \text{Sinh}\,\lambda x \cos \lambda x')]. \tag{48b}$$

Slopes at the end points:

$$\theta_A = \frac{2M_0\lambda^3}{k}\frac{\text{Cosh}\,\lambda l\,\text{Sinh}\,\lambda l - \cos\lambda l\sin\lambda l}{\text{Cosh}^2\lambda l - \cos^2\lambda l},$$

$$\theta_B = -\frac{2M_0\lambda^3}{k}\frac{\text{Cosh}\,\lambda l\sin\lambda l - \text{Sinh}\,\lambda l\cos\lambda l}{\text{Cosh}^2\lambda l - \cos^2\lambda l}.$$

$$M = M_0\frac{1}{\text{Cosh}^2\lambda l - \cos^2\lambda l}(\text{Cosh}\,\lambda l\cos\lambda x\,\text{Cosh}\,\lambda x' - \cos\lambda l\,\text{Cosh}\,\lambda x\cos\lambda x'). \quad (48c)$$

$$Q = -M_0\lambda\frac{1}{\text{Cosh}^2\lambda l - \cos^2\lambda l}[\text{Cosh}\,\lambda l(\cos\lambda x\,\text{Sinh}\,\lambda x' + \sin\lambda x\,\text{Cosh}\,\lambda x') + \cos\lambda l(\text{Sinh}\,\lambda x\cos\lambda x' + \text{Cosh}\,\lambda x\sin\lambda x')]. \quad (48d)$$

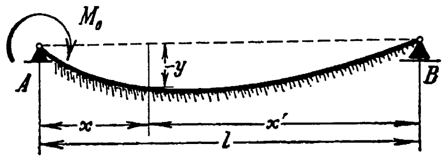

FIG. 53

Hence the expressions for the reaction forces are

$$A = -M_0\lambda\frac{\text{Cosh}\,\lambda l\,\text{Sinh}\,\lambda l + \cos\lambda l\sin\lambda l}{\text{Cosh}^2\lambda l - \cos^2\lambda l},$$

$$B = M_0\lambda\frac{\text{Cosh}\,\lambda l\sin\lambda l + \text{Sinh}\,\lambda l\cos\lambda l}{\text{Cosh}^2\lambda l - \cos^2\lambda l}.$$

21. *Beams with Fixed Ends*

a. Concentrated Force in the Middle (Fig. 54)

$$y = \frac{P\lambda}{2k}\frac{1}{\text{Sinh}\,\lambda l + \sin\lambda l}\{\sin\lambda x\,\text{Sinh}\,\lambda(l - x) - \text{Sinh}\,\lambda x\sin\lambda(l - x) - \cos\lambda x[\text{Cosh}\,\lambda x - \text{Cosh}\,\lambda(l - x)] - \text{Cosh}\,\lambda x[\cos\lambda x - \cos\lambda(l - x)]\}. \quad (49a)$$

Deflection at the middle:

$$y_C = \frac{P\lambda}{2k}\frac{\text{Cosh}\,\lambda l + \cos\lambda l - 2}{\text{Sinh}\,\lambda l + \sin\lambda l}.$$

$$\theta = \frac{P\lambda^2}{k}\frac{\sin\lambda x[\text{Cosh}\,\lambda x - \text{Cosh}\,\lambda(l - x)] - \text{Sinh}\,\lambda x[\cos\lambda x - \cos\lambda(l - x)]}{\text{Sinh}\,\lambda l + \sin\lambda l}. \quad (49b)$$

$$M = -\frac{P}{4\lambda}\frac{1}{\text{Sinh}\,\lambda l + \sin\lambda l}\{\sin\lambda x[\text{Sinh}\,\lambda x + \text{Sinh}\,\lambda(l - x)] + \text{Sinh}\,\lambda x[\sin\lambda x + \sin\lambda(l - x)] - \cos\lambda x\,\text{Cosh}\,\lambda(l - x) + \text{Cosh}\,\lambda x\cos\lambda(l - x)\}. \quad (49c)$$

Particular values of M:

$$M_A = M_B = -\frac{P}{\lambda} \frac{\operatorname{Sinh} \frac{\lambda l}{2} \sin \frac{\lambda l}{2}}{\operatorname{Sinh} \lambda l + \sin \lambda l},$$

$$M_C = \frac{P}{4\lambda} \frac{\operatorname{Cosh} \lambda l - \cos \lambda l}{\operatorname{Sinh} \lambda l + \sin \lambda l}.$$

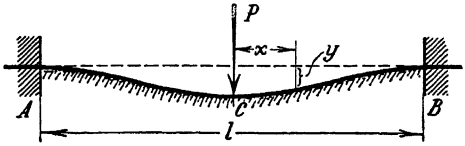

FIG. 54

$$Q = -\frac{P}{2} \frac{1}{\operatorname{Sinh} \lambda l + \sin \lambda l} \{\operatorname{Cosh} \lambda x[\sin \lambda x + \sin \lambda(l - x)] + \cos \lambda x[\operatorname{Sinh} \lambda x + \operatorname{Sinh} \lambda(l - x)]\}. \quad (49d)$$

Hence the expression for the reactions is:

$$A = B = P \frac{\operatorname{Cosh} \frac{\lambda l}{2} \sin \frac{\lambda l}{2} + \cos \frac{\lambda l}{2} \operatorname{Sinh} \frac{\lambda l}{2}}{\operatorname{Sinh} \lambda l + \sin \lambda l}.$$

b. Uniformly Distributed Loading over the Whole Span (Fig. 55)

$$y = \frac{q}{k}\left[1 - \frac{1}{\operatorname{Sinh} \lambda l + \sin \lambda l} (\operatorname{Sinh} \lambda x \cos \lambda x' + \sin \lambda x \operatorname{Cosh} \lambda x' + \operatorname{Sinh} \lambda x' \cos \lambda x + \sin \lambda x' \operatorname{Cosh} \lambda x)\right]. \quad (50a)$$

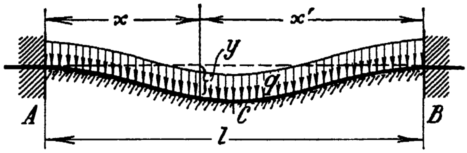

FIG. 55

Deflection at the middle:

$$y_C = \frac{q}{k}\left[1 - \frac{2\left(\operatorname{Sinh} \frac{\lambda l}{2} \cos \frac{\lambda l}{2} + \operatorname{Cosh} \frac{\lambda l}{2} \sin \frac{\lambda l}{2}\right)}{\operatorname{Sinh} \lambda l + \sin \lambda l}\right].$$

$$\theta = -\frac{2q\lambda}{k} \frac{\operatorname{Sinh} \lambda x \sin \lambda x' - \sin \lambda x \operatorname{Sinh} \lambda x'}{\operatorname{Sinh} \lambda l + \sin \lambda l}. \quad (50b)$$

$$M = -\frac{q}{2\lambda^2} \frac{1}{\operatorname{Sinh} \lambda l + \sin \lambda l} (\operatorname{Sinh} \lambda x \cos \lambda x' + \cos \lambda x \operatorname{Sinh} \lambda x' - \sin \lambda x \operatorname{Cosh} \lambda x' - \operatorname{Cosh} \lambda x \sin \lambda x'). \quad (50c)$$

Particular values of M:

$$M_A = M_B = -\frac{q}{2\lambda^2} \frac{\operatorname{Sinh} \lambda l - \sin \lambda l}{\operatorname{Sinh} \lambda l + \sin \lambda l},$$

$$M_C = \frac{q}{\lambda^2} \frac{\sin \frac{\lambda l}{2} \operatorname{Cosh} \frac{\lambda l}{2} - \cos \frac{\lambda l}{2} \operatorname{Sinh} \frac{\lambda l}{2}}{\operatorname{Sinh} \lambda l + \sin \lambda l}.$$

$$Q = -\frac{q}{\lambda} \frac{\operatorname{Cosh} \lambda x \cos \lambda x' - \cos \lambda x \operatorname{Cosh} \lambda x'}{\operatorname{Sinh} \lambda l + \sin \lambda l}. \quad (50d)$$

Hence the expression for the reactions is:

$$A = B = \frac{q}{\lambda}\frac{\text{Cosh}\,\lambda l - \cos\lambda l}{\text{Sinh}\,\lambda l + \sin\lambda l}.$$

22. *Cantilever Beams*

a. *Concentrated Force at the End (Fig. 56)*

$$y = \frac{2P\lambda}{k}\frac{\text{Sinh}\,\lambda x\cos\lambda x'\,\text{Cosh}\,\lambda l - \sin\lambda x\,\text{Cosh}\,\lambda x'\cos\lambda l}{\text{Cosh}^2\lambda l + \cos^2\lambda l}. \tag{51a}$$

Deflection at the end B:

$$y_B = \frac{P\lambda}{k}\frac{\text{Sinh}\,2\lambda l - \sin 2\lambda l}{\text{Cosh}^2\lambda l + \cos^2\lambda l}.$$

$$\theta = \frac{2P\lambda^2}{k}\frac{1}{\text{Cosh}^2\lambda l + \cos^2\lambda l}[\text{Cosh}\,\lambda l(\text{Cosh}\,\lambda x\cos\lambda x' + \text{Sinh}\,\lambda x\sin\lambda x') - \cos\lambda l(\cos\lambda x\,\text{Cosh}\,\lambda x' - \sin\lambda x\,\text{Sinh}\,\lambda x')]. \tag{51b}$$

Slope at the end B:

$$\theta_B = \frac{2P\lambda^2}{k}\frac{\text{Cosh}^2\lambda l - \cos^2\lambda l}{\text{Cosh}^2\lambda l + \cos^2\lambda l}.$$

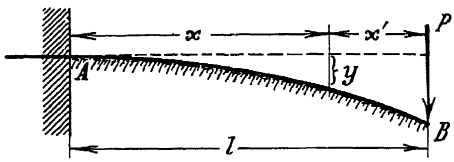

FIG. 56

$$M = -\frac{P}{\lambda}\frac{\text{Cosh}\,\lambda x\sin\lambda x'\,\text{Cosh}\,\lambda l + \cos\lambda x\,\text{Sinh}\,\lambda x'\cos\lambda l}{\text{Cosh}^2\lambda l + \cos^2\lambda l}. \tag{51c}$$

Bending moment at the end A:

$$M_A = -\frac{P}{\lambda}\frac{\text{Sinh}\,\lambda l\cos\lambda l + \text{Cosh}\,\lambda l\sin\lambda l}{\text{Cosh}^2\lambda l + \cos^2\lambda l}.$$

$$Q = -\frac{P}{\text{Cosh}^2\lambda l + \cos^2\lambda l}[\text{Cosh}\,\lambda l(\text{Sinh}\,\lambda x\sin\lambda x' - \text{Cosh}\,\lambda x\cos\lambda x') - \cos\lambda l(\sin\lambda x\,\text{Sinh}\,\lambda x' + \cos\lambda x\,\text{Cosh}\,\lambda x')]. \tag{51d}$$

Reaction force at the end A:

$$Q_A = P\frac{2\,\text{Cosh}\,\lambda l\cos\lambda l}{\text{Cosh}^2\lambda l + \cos^2\lambda l}.$$

b. *Uniformly Distributed Loading over the Whole Span (Fig. 57)*

$$y = \frac{q}{k}\left\{1 - \frac{1}{\text{Cosh}^2\lambda l + \cos^2\lambda l}[\text{Cosh}\,\lambda l(\sin\lambda x\,\text{Sinh}\,\lambda x' + \cos\lambda x\,\text{Cosh}\,\lambda x') - \cos\lambda l(\text{Sinh}\,\lambda x\sin\lambda x' - \text{Cosh}\,\lambda x\cos\lambda x')]\right\}. \tag{52a}$$

Deflection at the end B:

$$y_B = \frac{q}{k}\left(1 - \frac{2\,\text{Cosh}\,\lambda l \cos \lambda l}{\text{Cosh}^2 \lambda l + \cos^2 \lambda l}\right).$$

$$\theta = -\frac{2q\lambda}{k}\,\frac{\text{Sinh}\,\lambda x \cos \lambda x' \cos \lambda l - \sin \lambda x\,\text{Cosh}\,\lambda x'\,\text{Cosh}\,\lambda l}{\text{Cosh}^2 \lambda l + \cos^2 \lambda l}. \quad (52b)$$

Slope at the end B:

$$\theta_B = -\frac{2q\lambda}{k}\,\frac{\text{Sinh}\,\lambda l \cos \lambda l - \text{Cosh}\,\lambda l \sin \lambda l}{\text{Cosh}^2 \lambda l + \cos^2 \lambda l}.$$

$$M = \frac{q}{2\lambda^2}\,\frac{1}{\text{Cosh}^2 \lambda l + \cos^2 \lambda l}\,[\text{Cosh}\,\lambda l(\sin \lambda x\,\text{Sinh}\,\lambda x' - \cos \lambda x\,\text{Cosh}\,\lambda x')$$
$$+ \cos \lambda l(\text{Sinh}\,\lambda x \sin \lambda x' + \text{Cosh}\,\lambda x \cos \lambda x')]. \quad (52c)$$

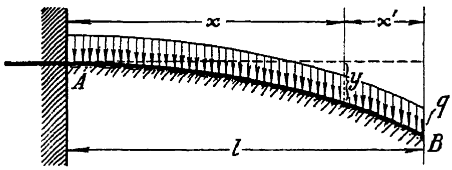

FIG. 57

Bending moment at the end A:

$$M_A = -\frac{q}{2\lambda^2}\,\frac{\text{Cosh}^2 \lambda l - \cos^2 \lambda l}{\text{Cosh}^2 \lambda l + \cos^2 \lambda l}.$$

$$Q = \frac{q}{\lambda}\,\frac{\text{Cosh}\,\lambda x \sin \lambda x' \cos \lambda l + \cos \lambda x\,\text{Sinh}\,\lambda x'\,\text{Cosh}\,\lambda l}{\text{Cosh}^2 \lambda l + \cos^2 \lambda l}. \quad (52d)$$

Reaction force at the end A:

$$Q_A = \frac{q}{2\lambda}\,\frac{\text{Sinh}\,2\lambda l + \sin 2\lambda l}{\text{Cosh}^2 \lambda l + \cos^2 \lambda l}.$$

c. Triangular Distributed Loading over the Whole Span (Fig. 58)

$$y = \frac{q_0}{k}\left(1 - \frac{x}{l}\right) + \frac{q_0}{2\lambda l k}\,\frac{1}{\text{Cosh}^2 \lambda l + \cos^2 \lambda l}\,\{\text{Sinh}\,\lambda x[\cos \lambda x + \cos \lambda(2l - x)]$$
$$+ \sin \lambda x[\text{Cosh}\,\lambda x + \text{Cosh}\,\lambda(2l - x)]$$
$$+ 2\lambda l[\cos \lambda(l - x)\cdot(\text{Sinh}\,\lambda x \sin \lambda l - \text{Cosh}\,\lambda x \cos \lambda l)$$
$$- \text{Cosh}\,\lambda(l - x)\cdot(\text{Sinh}\,\lambda l \sin \lambda x + \text{Cosh}\,\lambda l \cos \lambda x)]\}. \quad (53a)$$

$$\theta = -\frac{q_0}{kl} + \frac{q_0}{2kl}\,\frac{1}{\text{Cosh}^2 \lambda l + \cos^2 \lambda l}\,\{\text{Cosh}\,\lambda x[\cos \lambda x + \cos \lambda(2l - x)]$$
$$+ \cos \lambda x[\text{Cosh}\,\lambda x + \text{Cosh}\,\lambda(2l - x)]$$
$$+ \text{Sinh}\,\lambda x \sin \lambda(2l - x) - \sin \lambda x\,\text{Sinh}\,\lambda(2l - x)$$
$$+ 2\lambda l[\sin \lambda x(\text{Cosh}\,\lambda x + \text{Cosh}\,\lambda(2l - x))$$
$$- \text{Sinh}\,\lambda x(\cos \lambda x + \cos \lambda(2l - x))]\}. \quad (53b)$$

$$M = -\frac{q_0}{4\lambda^3 l}\frac{1}{\text{Cosh}^2 \lambda l + \cos^2 \lambda l}\{\cos \lambda x[\text{Sinh}\, \lambda x - \text{Sinh}\, \lambda(2l - x)] - \text{Cosh}\, \lambda x[\sin \lambda x - \sin \lambda(2l - x)] + 2\lambda l[\text{Sinh}\, \lambda(l - x)\cdot(\text{Sinh}\, \lambda l \cos \lambda x - \text{Cosh}\, \lambda l \sin \lambda x) + \sin \lambda(l - x)\cdot(\text{Cosh}\, \lambda x \sin \lambda l - \text{Sinh}\, \lambda x \cos \lambda l)]\}. \quad (53c)$$

Bending moment at the end A:

$$M_A = -\frac{q_0}{4\lambda^3 l}\frac{\sin 2\lambda l - \text{Sinh}\, 2\lambda l + 2\lambda l(\text{Sinh}^2 \lambda l + \sin^2 \lambda l)}{\text{Cosh}^2 \lambda l + \cos^2 \lambda l}.$$

$$Q = -\frac{q_0}{2\lambda^2 l}\frac{1}{\text{Cosh}^2 \lambda l + \cos^2 \lambda l}\Big[\text{Cosh}\, \lambda l[\sin \lambda x\, \text{Sinh}\, \lambda(l - x) + \cos \lambda x\, \text{Cosh}\, \lambda(l - x)] + \cos \lambda l[\text{Sinh}\, \lambda x \sin \lambda(l - x) - \text{Cosh}\, \lambda x \cos \lambda(l - x)] + \lambda l\{\text{Cosh}\, \lambda x[\sin \lambda x - \sin \lambda(2l - x)] + \cos \lambda x[\text{Sinh}\, \lambda x - \text{Sinh}\, \lambda(2l - x)]\}\Big]. \quad (53d)$$

Reaction force at the end A:

$$Q_A = \frac{q_0}{2\lambda^2 l}\frac{\lambda l(\text{Sinh}\, 2\lambda l + \sin 2\lambda l) - (\text{Sinh}^2 \lambda l + \sin^2 \lambda l)}{\text{Cosh}^2 \lambda l + \cos^2 \lambda l}.$$

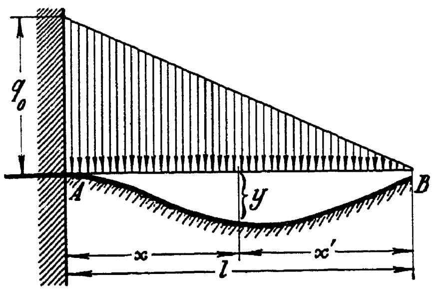

FIG. 58

From the formulas above simple approximate expressions can be derived for cases where λl is relatively large ($\lambda l > 5$). Such large values of λl are most frequent in the design of cylindrical tanks and containers, which constitute an important application of these formulas. It has been pointed out in §17, page 46, that when $\lambda l > 5$ the beam can be considered for practical purposes as one of unlimited length, when it is permissible to put $\text{Sinh}\, \lambda l = \text{Cosh}\, \lambda l = e^{\lambda l}$ and to neglect the $\sin \lambda l$ and $\cos \lambda l$ terms in comparison with the hyperbolic functions of the same argument. With these simplifications expressions (53 a–d) will take the following form:

$$\left.\begin{aligned} y &= \frac{q_0}{\lambda l k}[\lambda(l - x) - \lambda l A_{\lambda x} + B_{\lambda x}],\\ \theta &= -\frac{q_0}{lk}(1 - 2\lambda l B_{\lambda x} - C_{\lambda x}),\\ M &= -\frac{q_0}{2\lambda^3 l}(\lambda l C_{\lambda x} - D_{\lambda x}),\\ Q &= \frac{q_0}{2\lambda^2 l}(2\lambda l D_{\lambda x} - A_{\lambda x}). \end{aligned}\right\} \quad (54\text{ a–d})$$

Here the same notations were used as in Chapter II (p. 12):

$$A_{\lambda x} = D_{\lambda x} + B_{\lambda x}, \qquad C_{\lambda x} = D_{\lambda x} - B_{\lambda x},$$

where

$$B_{\lambda x} = e^{-\lambda x} \sin \lambda x \quad \text{and} \quad D_{\lambda x} = e^{-\lambda x} \cos \lambda x.$$

At point A we shall now have the following values:

$$M_A = -\frac{q_0}{2\lambda^3 l}(\lambda l - 1),$$

$$Q_A = \frac{q_0}{2\lambda^2 l}(2\lambda l - 1).$$

23. *Partially Supported Beams*

a. Concentrated Force at the Middle of a Free Span (Fig. 59)

For points A and B (see Fig. 59) respectively the following values can be derived:

$$\left.\begin{aligned} M_A &= M_B = -\frac{P}{8\lambda}\,\frac{\lambda^2 L^2(\text{Sinh}^2 \lambda l - \sin^2 \lambda l) - 4(\text{Sinh}^2 \lambda l + \sin^2 \lambda l)}{\lambda L(\text{Sinh}^2 \lambda l - \sin^2 \lambda l) + \text{Sinh}\, 2\lambda l + \sin 2\lambda l}, \\ Q_A &= -Q_B = \frac{1}{2}P. \end{aligned}\right\} \tag{a}$$

Putting these values of Q and M in place of P and M_0 into (39 a–d) and (40 a–d), we can obtain the complete solution for the side portions A–C and B–D respectively. For cases where $\lambda l > \pi$ the following approximate formulas can be used for the side spans:

$$\left.\begin{aligned} y &= \frac{P\lambda}{4k}[4D_{\lambda x} - (2 - \lambda L)C_{\lambda x}], \\ \theta &= \pm\frac{P\lambda^2}{2k}[2A_{\lambda x} - (2 - \lambda L)D_{\lambda x}], \\ M &= -\frac{P}{8\lambda}[4B_{\lambda x} - (2 - \lambda L)A_{\lambda x}], \\ Q &= \pm\frac{P}{4}[2C_{\lambda x} - (2 - \lambda L)B_{\lambda x}], \end{aligned}\right\} \tag{55 a–d}$$

where x is counted from the end of the free span, as shown in Figure 59.

For points A and B we have now the following values:

$$y_A = y_B = \frac{P\lambda}{4k}(2 + \lambda L),$$

$$\theta_A = -\theta_B = \frac{P\lambda^3 L}{2k},$$

$$M_A = M_B = \frac{P}{8\lambda}(2 - \lambda L),$$

$$Q_A = -Q_B = \frac{P}{2}.$$

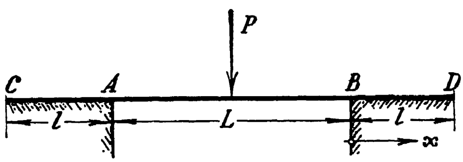

FIG. 59

It is seen that $M_A = M_B = 0$ when $\lambda L = 2$, and that M_A and M_B are positive if $\lambda L < 2$ and negative if $\lambda L > 2$.

b. Uniformly Distributed Loading over a Free Span (Fig. 60)

For points A and B (see Fig. 60) respectively the following values can be derived:

$$\left.\begin{aligned} M_A &= M_B = -\frac{qL}{12\lambda}\,\frac{\lambda^2L^2(\text{Sinh}^2\,\lambda l - \sin^2\lambda l) - 6(\text{Sinh}^2\,\lambda l + \sin^2\lambda l)}{\lambda L(\text{Sinh}^2\,\lambda l - \sin^2\lambda l) + \text{Sinh}\,2\lambda l + \sin 2\lambda l}, \\ Q_A &= -Q_B = \frac{qL}{2}. \end{aligned}\right\} \quad \text{(b)}$$

Substituting these values of Q and M in place of P and M_0 in (39 a–d) and (40 a–d), we can obtain complete solutions for the side spans A–C and B–D respectively. If $\lambda l > \pi$, the following approximate formulas can be used for the side spans:

$$\left.\begin{aligned} y &= \frac{q\lambda L}{k}(D_{\lambda x} - \alpha C_{\lambda x}), \\ \theta &= \pm\frac{q\lambda^2 L}{k}(A_{\lambda x} - 2\alpha D_{\lambda x}), \\ M &= -\frac{qL}{2\lambda}(B_{\lambda x} - \alpha A_{\lambda x}), \\ Q &= \pm\frac{qL}{2}(C_{\lambda x} + 2\alpha B_{\lambda x}), \end{aligned}\right\} \quad \text{(56 a–d)}$$

where

$$\alpha = \frac{6 - \lambda^2L^2}{6(2 + \lambda L)}$$

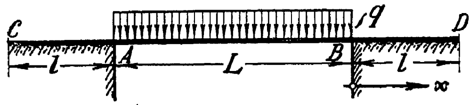

FIG. 60

and x is measured from the end of the free span, as shown in Figure 60.

For points A and B we have now the following values:

$$\begin{aligned} y_A &= y_B = \frac{q\lambda L}{k}(1 - \alpha), \\ \theta_A &= -\theta_B = \frac{q\lambda^2 L}{k}(1 - 2\alpha), \\ M_A &= M_B = \frac{qL}{2\lambda}\alpha, \\ Q_A &= -Q_B = \frac{qL}{2}. \end{aligned}$$

It is seen that $M_A = M_B = 0$ if $\lambda L = \sqrt{6} = 2.45$, and that M_A and M_B are positive if $\lambda L < \sqrt{6}$ and negative if $\lambda L > \sqrt{6}$.

If the portions to the left of point A and to the right of point B are also loaded with a uniformly distributed load (let it be p), then we must add a $y_0 = p/k$ value to the expression of y in (56a), while the values of θ, M, and Q in (56 b–d) will remain unchanged.

II. Solutions in the Form of Trigonometric Series

24. *Beams with Free Ends*

In the literature on bending of beams* frequent use has been made of the fact that any elastic line which passes through the points $x = 0, y = 0$, and $x = l$, $y = 0$ can be represented in the form of a trigonometric series, as

$$y = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l}.$$

This series, with some modifications, can also be applied in the analysis of beams on elastic foundation.

Taking a beam with free ends, supported on an elastic foundation and subjected to loading, we find that the conditions above, concerning the deflections at the ends, are generally not fulfilled. Displacements will occur at both ends and the deflection diagram will look like that in Figure 61. In order to be able to apply the series method to problems of this type, the deflection diagram in Figure 61 will be regarded as having originated in three steps. We may suppose (1) that the whole beam was displaced, so that it was parallel with itself, by a constant y_0; (2) that it was rotated by an angle α around its center; and (3) that it underwent elastic deformation, as shown by the $A''C''B''$ curve in Figure 61. Since this elastic line will have zero ordinates at $x = 0$ and $x = l$, it will always be possible to express it by a sine series like the one in the equation above.

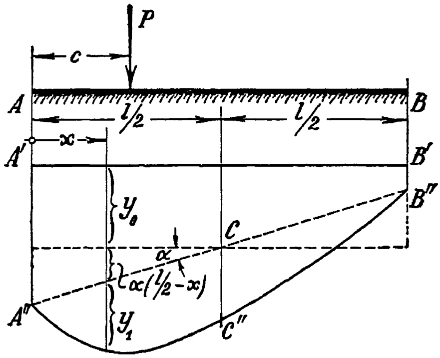

FIG. 61

Adding together the three deflection components, we obtain for any deflection ordinate the expression

$$y = y_0 + \alpha\left(\frac{l}{2} - x\right) + \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l}. \tag{57}$$

In order to simplify the solution of this general case we shall split it into two parts, the symmetrical and the antisymmetrical loading components, as was done before.

* See S. Timoshenko, *Strength of Materials* (2d ed.; New York, 1941), Part II, p. 44.

a. Symmetrical Case (Figs. 62–63)

From the symmetry of the deflection curve we can conclude at once that for this case we shall have in (57) the term $\alpha = 0$, and, furthermore, that the sine series will contain now only odd terms. Hence the expression for the deflection curve will be

$$y = y_0 + \sum_{n=1,3,5,\ldots}^{\infty} a_n \sin \frac{n\pi x}{l}. \tag{a}$$

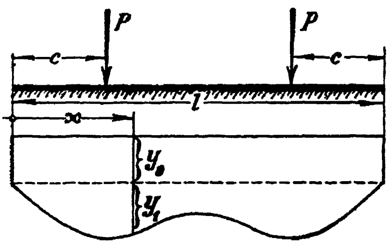

FIG. 62

Denoting the modulus of the foundation by k_0 and the width of the beam by b, and putting $k_0 b = k$, we know that the distributed reaction p caused in the supporting medium by a deflection y will be

$$p = ky.$$

Assuming that the beam is loaded with two symmetrically placed concentrated forces as shown in Figure 62, according to the conditions of equilibrium we must have

$$2P = k\int_0^l y\,dx = k\int_0^l \left(y_0 + \sum_{n=1,3,5,\ldots}^{\infty} a_n \sin\frac{n\pi x}{l}\right)dx$$

$$= k\left(y_0 l + \frac{2l}{\pi}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n}a_n\right).$$

From this we have

$$y_0 = \frac{2P}{kl} - \frac{2}{\pi}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n}a_n. \tag{b}$$

We shall determine the coefficients a_n in (b) by considering the strain energy of the whole system. The strain energy of bending of the beam will be

$$V_1 = \frac{1}{2EI}\int_0^l M^2\,dx = \frac{EI}{2}\int_0^l y''^2\,dx = \frac{\pi^4 EI}{4l^3}\sum_{n=1,3,5,\ldots}^{\infty} n^4 a_n^2. \tag{c}$$

The strain energy of deformation in the foundation will be

$$V_2 = \tfrac{1}{2}k\int_0^l y^2\,dx = \tfrac{1}{2}k\int_0^l\left[\frac{2P}{kl} - \sum_{n=1,3,5,\ldots}^{\infty} a_n\left(\frac{2}{n\pi} - \sin\frac{n\pi x}{l}\right)\right]^2 dx. \tag{d}$$

Since the whole system is in equilibrium, the work done by the external forces by any small change da_n in the elastic line must, according to the principle of virtual displacements, equal the corresponding variation in the strain energy of the system, that is,

$$2P\frac{\partial y_P}{\partial a_n}da_n = \frac{\partial V_1}{\partial a_n}da_n + \frac{\partial V_2}{\partial a_n}da_n. \tag{e}$$

Determining each term in this expression from equations (a)–(d), we can write (e) in the form

$$2P\left(\sin\frac{n\pi c}{l} - \frac{2}{n\pi}\right) = a_n\left(n^4\frac{\pi^4 EI}{2l^3} + \frac{lk}{2}\right) - \frac{4lk}{\pi^2}\frac{1}{n}\sum_{i=1,3,5,\cdots}^{\infty}\frac{1}{i}a_i. \qquad (58)$$

With the aid of this formula we can determine as many a terms as are necessary for the required accuracy in the deflection line. In every case we shall have a system of linear equations with the a_n terms as unknowns. Putting numbers instead of the symbols n and i into (58), we get the scheme

$$2P\left(\sin\frac{\pi c}{l} - \frac{2}{\pi}\right) = a_1\left(\frac{\pi^4 EI}{2l^3} + \frac{lk}{2}\right) - \frac{4lk}{\pi^2}\left(a_1 + \tfrac{1}{3}a_3 + \tfrac{1}{5}a_5 + \cdots\right),$$

$$2P\left(\sin\frac{3\pi c}{l} - \frac{2}{3\pi}\right) = a_3\left(3^4\frac{\pi^4 EI}{2l^3} + \frac{lk}{2}\right) - \frac{4lk}{\pi^2}\tfrac{1}{3}\left(a_1 + \tfrac{1}{3}a_3 + \tfrac{1}{5}a_5 + \cdots\right),$$

$$2P\left(\sin\frac{5\pi c}{l} - \frac{2}{5\pi}\right) = a_5\left(5^4\frac{\pi^4 EI}{2l^3} + \frac{lk}{2}\right) - \frac{4lk}{\pi^2}\tfrac{1}{5}\left(a_1 + \tfrac{1}{3}a_3 + \tfrac{1}{5}a_5 + \cdots\right).$$

From this we can determine the coefficients a_1, a_3, a_5, and from (b) we can obtain the quantity y_0. Putting the latter into (a) we have the formula for the deflection line:

$$y = \frac{2P}{kl} - \sum_{n=1,3,5,\cdots}^{\infty} a_n\left(\frac{2}{n\pi} - \sin\frac{n\pi x}{l}\right), \qquad (59)$$

and this is completely defined by the a terms obtained above.

The system of equations in (58) is such as can readily be solved by the method of successive approximations.

When the beam is loaded at the center or at the ends ($c = l/2$ or $c = 0$) good approximation of the deflection line can be obtained by taking only the first term (a_1) from the series in (58), and thus we get:

$$a_1 = \frac{2P\left(\sin\frac{\pi c}{l} - \frac{2}{\pi}\right)}{\frac{\pi^4 EI}{2l^3} + \frac{lk}{2} - \frac{4lk}{\pi^2}}.$$

Substituting this in (59), we get

$$y = \frac{2P}{kl} - a_1\left(\frac{2}{\pi} - \sin\frac{\pi x}{l}\right). \qquad (60)$$

The slope of the deflection line can be obtained in any case by differentiating the general equation (59) with respect to x. Instead of taking the higher derivatives of the elastic line, we can derive expressions for shearing forces and moments in a more accurate way by the following reasoning. The beam can be regarded as loaded not only by the external forces, but also by the distributed reaction $p = ky$ of the foundation, which keeps equilibrium with the external loading. Thus, knowing all the forces acting on the beam, we can obtain the Q and M curves by integrating (59). This integration can be carried out numerically or graphically. Application of trigonometric series is sometimes economical in determining deflection curves (for instance, in dealing with railroad ties), but we must do considerable additional computing when M and Q curves are required. Therefore, if a complete analysis is needed, it is usually simpler to use directly the exact analysis presented in the preceding sections. For this reason we will not discuss at this place other cases of loadings besides that involving two symmetrically placed equal concentrated forces. For more complex loadings the application of this series method is not economical because too many terms have to be computed in order to define the elastic curve accurately.

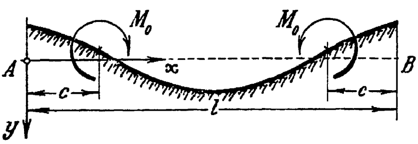

FIG. 63

From the solution above for concentrated forces we can derive an approximate expression for the case when the beam is loaded with concentrated moments (Fig. 63). Using only one term of the series, as in (60), we first take two downward-acting forces P at a distance c' from the ends and then two forces P of the same magnitude acting upward at a distance c'' from the ends. Superposing the two on each other, we have

$$a_1 = \frac{2P\left(\sin\frac{\pi c'}{l} - \frac{2}{\pi} - \sin\frac{\pi c''}{l} + \frac{2}{\pi}\right)}{\frac{\pi^4 EI}{2l^3} + \frac{lk}{2} - \frac{4lk}{\pi^2}}.$$

Taking the limit when the two forces approach each other ($c' \to c''$ and $c'' \to c$) and at the same time putting $P(c' - c'') = M_0$, we have

$$a_1 = \frac{M_0 \frac{2\pi}{l} \cos\frac{\pi c}{l}}{\frac{\pi^4 EI}{2l^3} + \frac{lk}{2} - \frac{4lk}{\pi^2}};$$

and thus the approximate expression for the deflection curve of the beam in Figure 63, since $2P/kl = 0$, will be

$$y = -a_1\left(\frac{2}{\pi} - \sin\frac{\pi x}{l}\right). \tag{61}$$

b. Antisymmetrical Case (Figs. 64–65)

The beam will now have a deflection curve of the form shown in Figure 64. We can deduce at once that here $y_0 = 0$, and that the series will include only even terms. Hence we can write the deflection curve as

$$y = \alpha\left(\frac{l}{2} - x\right) + \sum_{n=2,4,6,\cdots}^{\infty} a_n \sin\frac{n\pi x}{l}. \tag{f}$$

Considering the equilibrium of moments, for the type of loading shown in Figure 64, we have the relation

$$P(l - 2c) + k \int_0^l yx\,dx = 0. \tag{g}$$

Putting in (g) the expression for y from (f) and carrying out the integration, we get

$$\alpha = \frac{12}{l^3}\left[\frac{P(l - 2c)}{k} - \frac{l^2}{\pi} \sum_{n=2,4,6,\dots}^{\infty} a_n \sin \frac{n\pi x}{l}\right]. \tag{h}$$

Substituting this in (f) we have the equation for the deflection curve in Figure 64 as

$$y = \frac{12P(l - 2c)}{kl^3}\left(\frac{l}{2} - x\right) - \left[\frac{12}{l\pi}\left(\frac{l}{2} - x\right) - 1\right] \sum_{n=2,4,6,\dots}^{\infty} a_n \sin \frac{n\pi x}{l}. \tag{62}$$

The a_n terms can be determined in the same way as in the symmetrical case, by equating the change in the strain energy of bending of the bar and in that of the deformation in the foundation to the external work done by the loading during a virtual displacement. In this way we obtain for the present problem the general formula

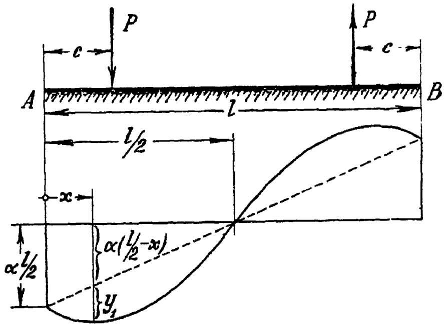

FIG. 64

$$2P\left[\sin \frac{n\pi c}{l} - \frac{6}{n\pi l}(l - 2c)\right] = a_n\left(n^4 \frac{\pi^4 EI}{2l^3} + \tfrac{1}{2}kl\right) - \frac{12kl}{n\pi^2} \sum_{i=2,4,6,\dots}^{\infty} \frac{1}{i} a_i, \tag{63}$$

from which, as from (58), any number of a_n terms can be computed.

In some cases satisfactory approximation is obtained by taking only the first term ($n = 2$), from (63), getting thus

$$a_2 = \frac{2P\left(\sin \frac{2\pi c}{l} - \frac{3}{l\pi}(l - 2c)\right)}{16 \frac{\pi^4 EI}{2l^3} + \tfrac{1}{2}kl - \frac{3kl}{\pi^2}}.$$

The shearing-force and bending-moment curves can be obtained most easily by successive integration of the deflection curve, as has previously been suggested.

By making use of the first term (a_2) alone an approximate expression can be derived also for the case when the beam is subjected to two concentrated moments, as shown in Figure 65. Taking forces P at distances c' from the ends and forces $-P$ at distances c'' from the ends, the limiting case of $c' \to c'' \to c$ and $P(c' - c'') \to M_0$ leads to the formula

$$a_2 = \frac{M_0 \frac{4\pi}{l} \cos \frac{2\pi c}{l}}{16 \frac{\pi^4 EI}{2l^3} + \frac{kl}{2} - \frac{3kl}{\pi^2}},$$

which then gives the approximate deflection curve as

$$y = -a_2 \left[\frac{6}{\pi l} \left(\frac{l}{2} - x \right) - \sin \frac{2\pi x}{l} \right]. \tag{64}$$

It is seen that the application of trigonometric series, so far as beams with free ends are concerned, becomes rather cumbersome and ordinarily does not seem to offer any advantage over the previous solutions based on the differential equation of the elastic line. Sometimes, however, just the first term of the series solution will give a good approximation for the deflection curve of the beam, and in such cases the use of the method becomes profitable. In order to examine this situation, let us find the necessary length of beam which, when loaded with a concentrated force in the middle, will exhibit no deflections at the ends ($y_0 = 0$).

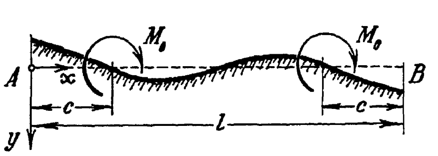

FIG 65

From (60), putting $c = l/2$ and $x = 0$, we have

$$y_0 = \frac{2P}{kl} - \frac{2}{\pi} \frac{2P\left(1 - \frac{2}{\pi}\right)}{\frac{\pi^4 EI}{2l^3} + \frac{kl}{2} - \frac{4kl}{\pi^2}}.$$

From this expression we obtain the required length as $l = 3.072 \, (\sqrt[4]{4EI/k})$ or, in another form, $\lambda l = 3.072$. The exact answer to this problem (the effective length) is, as was pointed out on page 54, $\lambda l = \pi = 3.142$, the difference being 2.3 per cent. According to the reciprocity theorem, this value $l = \pi/\lambda$ will also be the length of a beam which, when loaded with equal concentrated forces at the ends, will have no deflection at the middle.

The value $\lambda l = \pi$ is close to the upper limit for beams of medium length,* for which the deflection analysis has the greatest importance. Thus we can say that the first term of the series alone will give good approximate results over the whole range of beams of medium length, providing the beam is loaded by concentrated forces at the ends or in the middle. These loadings, however, com-

* See § 17 on p. 46.

prise a great share of the practical problems for which the use of the series method is most convenient.

In the cases where only one term is necessary to give the required accuracy, the series method also has a particular advantage over the exact analysis, for, as is seen from the corresponding formulas, in the series method the dimensions of the beam (E, I, and l) and the modulus of the foundation (k_0 and $k = bk_0$) appear separately, and are not included in trigonometric and exponential functions through λ, as is true of the exact method. This feature of the series method makes it applicable in solving problems when one of the dimensions of the beam or of the foundation is unknown and must be determined in such a way as to satisfy some specific requirement set up regarding the elastic line. The procedure to follow in such *design problems* can be illustrated by the numerical example below.

Consider a beam having $E = 2 \times 10^6$ lbs./in.^{2}, $I = 32$ in.^{4}, and $k = bk_0 = 100$ lbs./in.^{2} and being loaded with a concentrated force P in the middle. Let us set the problem of finding what the length l of the beam must be at which the deflection at the middle will be twice as much as the deflection at the end, that is: $2y_0 = y_0 + a_1$ or $y_0 = a_1$. From (60) we can write this requirement in the form

$$\frac{2P}{kl} - \left(\frac{2}{\pi} + 1\right)a_1 = \frac{2P}{kl} - \frac{2P\left(1 - \frac{2}{\pi}\right)\left(1 + \frac{2}{\pi}\right)}{\frac{\pi^4 EI}{2l^3} + l\left(\frac{k}{2} - \frac{4k}{\pi^2}\right)} = 0,$$

which gives for the unknown length of the beam

$$l = \pi\sqrt[4]{\frac{EI}{k}} = 3.14\sqrt[4]{\frac{2 \times 10^6 \times 32}{100}} = 88.86 \text{ in.}$$

Taking this value for l, we have from (60) the deflection at the end as $y_0 = 0.138 \times 10^{-3}P$ and the deflection at the middle as $y_c = 0.276 \times 10^{-3}P$.

The exact formulas from (41a) give for this case

$$y_0 = y_A = 0.140 \times 10^{-3}P \quad \text{and} \quad y_c = 0.283 \times 10^{-3}P,$$

which are in the ratio $0.283/0.140 = 2.021$, differing only about 1 per cent from the required proportion.

25. *Beams with Hinged Ends*

The method of expressing the deflection line by trigonometric series is especially adapted to beams hinged at both ends. A sine series of the form

$$y = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l}$$

will satisfy completely the conditions of a hinged support ($y = 0$, $y'' = 0$) at both ends of the beam. Proceeding in the same way as in the previous section, we can determine the a_n terms of the series from the consideration of the strain energy of the system.

a. One Concentrated Force

Assuming the sine series form above for the deflection line, we find the strain energy of bending of the beam to be

$$V_1 = \frac{EI}{2} \int_0^l y''^2 \, dx = \frac{\pi^4 EI}{4l^3} \sum_{n=1}^{\infty} n^4 a_n^2 .$$

The strain energy of deformation in the foundation will be

$$V_2 = \frac{k}{2} \int_0^l y^2 \, dx = \frac{kl}{4} \sum_{n=1}^{\infty} a_n^2 .$$

If a small variation is produced in the a_n term, the resulting change in the strain energy will be

$$\frac{\partial V}{\partial a_n} da_n = \frac{\partial V_1}{\partial a_n} da_n + \frac{\partial V_2}{\partial a_n} da_n = \frac{\pi^4 EI}{2l^3} n^4 a_n \, da_n + \frac{kl}{2} a_n \, da_n .$$

At the same time, because of the change da_n, the assumed loading consisting of a force P at a distance c from the left end (Fig. 66) will do work of the amount

$$P \frac{\partial}{\partial a_n} y_P \, da_n = P \sin \frac{n\pi c}{l} da_n .$$

Equating this change in potential energy to the change in the strain energy of the system derived above, we have

$$P \sin \frac{n\pi c}{l} = \frac{\pi^4 EI}{2l^3} n^4 a_n + \frac{kl}{2} a_n ,$$

from which the formula for any a_n term can be obtained as

$$a_n = \frac{2Pl^3}{\pi^4 EI} \frac{\sin \frac{n\pi c}{l}}{n^4 + \frac{kl^4}{\pi^4 EI}} .$$

Consequently, the expression of the deflection curve for Figure 66 will then become

$$y = \frac{2Pl^3}{\pi^4 EI} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi c}{l} \sin \frac{n\pi x}{l}}{n^4 + \frac{kl^4}{\pi^4 EI}} . \tag{65}$$

If, instead of the symbolic notation for the series in (65), its individual terms are written out fully, we have the expression for y in the following form

$$y = \frac{2Pl^3}{\pi^4 EI} \left[\frac{\sin (\pi c/l) \sin (\pi x/l)}{1 + kl^4/\pi^4 EI} + \frac{\sin (2\pi c/l) \sin (2\pi x/l)}{16 + kl^4/\pi^4 EI} + \frac{\sin (3\pi c/l) \sin (3\pi x/l)}{81 + kl^4/\pi^4 EI} + \cdots \right].$$

The series is suitable in the form above for the computation of deflection lines; the convergence of the solution, however, can be substantially increased by the following reasoning.*

The hinged-end beam of Figure 66 can be regarded as acted upon by the concentrated force P and by the distributed reaction forces of the foundation $p = ky$, where y is to be taken from (65) above. Consequently, the final deflection of the beam can be resolved into two parts, the first (y^P) being produced by the force P, the second (y^p), by the distributed p reaction forces, both loadings acting on a beam simply supported at its ends, while the elastic foundation can be thought of as absent. The deflection y^P is obtained in a finite form by the elementary polynomial formula, and y^p can be determined by integrating four times the fundamental equation $EI(d^4y^p/dx^4) + ky = 0$, using for y the expression given by (65). In this way we obtain the deflection line for Figure 66 in the form

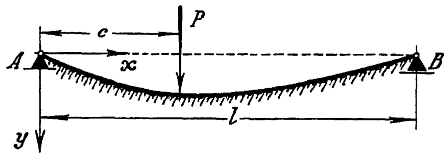

FIG. 66

$$y = y^P - \frac{2Pl^3}{\pi^4 EI} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi c}{l} \sin \frac{n\pi x}{l}}{n^4\left(1 + n^4 \frac{\pi^4 EI}{kl^4}\right)}. \tag{66a}$$

The series part of this expression is rapidly convergent, and its convergence will be good even after successive differentiation, which will yield the corresponding expressions for θ, M, and Q in the forms

$$\left.\begin{aligned} \theta &= \theta^P - \frac{2Pl^2}{\pi^3 EI} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi c}{l} \cos \frac{n\pi x}{l}}{n^3\left(1 + n^4 \frac{\pi^4 EI}{kl^4}\right)}, \\ M &= M^P - \frac{2Pl}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi c}{l} \sin \frac{n\pi x}{l}}{n^2\left(1 + n^4 \frac{\pi^4 EI}{kl^4}\right)}, \\ Q &= Q^P - \frac{2P}{\pi} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi c}{l} \cos \frac{n\pi x}{l}}{n\left(1 + n^4 \frac{\pi^4 EI}{kl^4}\right)}. \end{aligned}\right\} \tag{66 b–d}$$

In the expressions above, y^P, θ^P, M^P, and Q^P denote the deflection, slope, and so on produced by the concentrated force P on the beam with hinged ends, without the presence of an elastic foundation.

* See also the article by Th. von Kármán, "Use of Orthogonal Functions in Structural Problems," *Stephen Timoshenko 60th Anniversary Volume* (New York, 1938).

If $$x < c, \quad y^P = \frac{P}{6EI}\,\frac{l-c}{l}\,[(2l - c)cx - x^3];$$

if $$x > c, \quad y^P = \frac{P}{6EI}\,\frac{c}{l}\,[(l^2 - c^2)(l - x) - (l - x)^3].$$

b. One Concentrated Moment

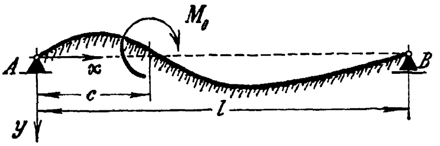

FIG. 67

From (65) of the previous section we can derive an expression for the deflection curve produced by a single concentrated bending moment M_0 (Fig. 67). Taking simultaneously a downward-acting force P at a distance c' and an upward-acting force $-P$ at a distance c'' from the left end of the beam, for the limiting case when $c' \to c$ and $c'' \to c$ we shall have

$$\sin\frac{n\pi c'}{l} - \sin\frac{n\pi c''}{l} = 2\,\frac{n\pi(c' - c'')}{2l}\cos\frac{n\pi c}{l}.$$

Taking at the same time $P(c' - c'') = M_0$, we obtain the deflection line due to the moment loading of Figure 67 as

$$y = \frac{2M_0 l^2}{\pi^3 EI}\sum_{n=1}^{\infty}\frac{n\cos\dfrac{n\pi c}{l}\sin\dfrac{n\pi x}{l}}{n^4 + \dfrac{kl^4}{\pi^4 EI}}. \tag{67}$$

The convergence of this formula can be increased in the same manner as in the loading with one concentrated force. This will lead now to the expressions

$$\left.\begin{aligned}
y &= y^M - \frac{2M_0 l^2}{\pi^3 EI}\sum_{n=1}^{\infty}\frac{\cos\dfrac{n\pi c}{l}\sin\dfrac{n\pi x}{l}}{n^3\left(1 + n^4\dfrac{\pi^4 EI}{kl^4}\right)},\\
\theta &= \theta^M - \frac{2M_0 l}{\pi^2 EI}\sum_{n=1}^{\infty}\frac{\cos\dfrac{n\pi c}{l}\cos\dfrac{n\pi x}{l}}{n^2\left(1 + n^4\dfrac{\pi^4 EI}{kl^4}\right)},\\
M &= M^M - \frac{2M_0}{\pi}\sum_{n=1}^{\infty}\frac{\cos\dfrac{n\pi c}{l}\sin\dfrac{n\pi x}{l}}{n\left(1 + n^4\dfrac{\pi^4 EI}{kl^4}\right)},\\
Q &= Q^M - \frac{2M_0}{l}\sum_{n=1}^{\infty}\frac{\cos\dfrac{n\pi c}{l}\cos\dfrac{n\pi x}{l}}{1 + n^4\dfrac{\pi^4 EI}{kl^4}},
\end{aligned}\right\} \tag{68 a–d}$$

where y^M, θ^M, M^M, and Q^M denote the deflection, slope, bending moment, and shearing force produced by the M_0 loading of Figure 67 on the beam with hinged ends, without the presence of an elastic foundation.

If $$x < c,\ y^M = \frac{M_0}{6EI}\frac{1}{l}\left[(2l^2 - 6lc + 3c^2)x + x^3\right];$$

if $$x > c,\ y^M = \frac{M_0}{6EI}\frac{1}{l}\left[(l^2 - 3c^2)(l - x) - (l - x)^3\right].$$

c. Uniformly Distributed Loading

For uniformly distributed loading we can again derive the deflection curve from (65), by putting $q\,dc$ instead of P and integrating between the limits c_1 and c_2 (Fig. 68), which gives

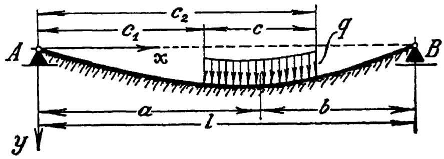

FIG. 68

$$q\int_{c_1}^{c_2} \sin\frac{n\pi c}{l}\,dc = \frac{ql}{n\pi}\left(\cos\frac{n\pi c_1}{l} - \cos\frac{n\pi c_2}{l}\right).$$

Substituting this expression instead of $P \sin n\pi c/l$ in (65), we have the deflection line for Figure 68 as

$$y = \frac{2ql^4}{\pi^5 EI}\sum_{n=1}^{\infty}\frac{\left(\cos\frac{n\pi c_1}{l} - \cos\frac{n\pi c_2}{l}\right)\sin\frac{n\pi x}{l}}{n\left(n^4 + \frac{kl^4}{\pi^4 EI}\right)}. \tag{69}$$

The convergence of this solution can be increased by the same reasoning used in the two previous types of loading, which will give now the complete solution for the situation in Figure 68 as

$$\left.\begin{aligned}
y &= y^q - \frac{2ql^4}{\pi^5 EI}\sum_{n=1}^{\infty}\frac{\left(\cos\frac{n\pi c_1}{l} - \cos\frac{n\pi c_2}{l}\right)\sin\frac{n\pi x}{l}}{n^5\left(1 + n^4\frac{\pi^4 EI}{kl^4}\right)},\\
\theta &= \theta^q - \frac{2ql^3}{\pi^4 EI}\sum_{n=1}^{\infty}\frac{\left(\cos\frac{n\pi c_1}{l} - \cos\frac{n\pi c_2}{l}\right)\cos\frac{n\pi x}{l}}{n^4\left(1 + n^4\frac{\pi^4 EI}{kl^4}\right)},\\
M &= M^q - \frac{2ql^2}{\pi^3}\sum_{n=1}^{\infty}\frac{\left(\cos\frac{n\pi c_1}{l} - \cos\frac{n\pi c_2}{l}\right)\sin\frac{n\pi x}{l}}{n^3\left(1 + n^4\frac{\pi^4 EI}{kl^4}\right)},\\
Q &= Q^q - \frac{2ql}{\pi^2}\sum_{n=1}^{\infty}\frac{\left(\cos\frac{n\pi c_1}{l} - \cos\frac{n\pi c_2}{l}\right)\cos\frac{n\pi x}{l}}{n^2\left(1 + n^4\frac{\pi^4 EI}{kl^4}\right)},
\end{aligned}\right\} \tag{70 a–d}$$

where y^q, θ^q, M^q, and Q^q denote quantities produced on the beam with hinged ends, without elastic foundation, by the q loading. With the notations $c_1 + c_2 = 2a$, $l - a = b$, $c_2 - c_1 = c$, shown in Figure 68, we have

$$\text{for} \qquad x < c_1, \quad y^q = \frac{q}{24EI}\frac{bc}{l}\{[4(l^2 - b^2) - c^2]x - 4x^3\},$$

$$\text{for} \quad c_1 < x < c_2, \quad y^q = \frac{q}{24EI}\frac{bc}{l}\left\{[4(l^2 - b^2) - c^2]x - 4x^3 + \frac{l(x - c_1)^4}{bc}\right\},$$

$$\text{for} \qquad x > c_2, \quad y^q = \frac{q}{24EI}\frac{ac}{l}\{[4(l^2 - a^2) - c^2](l - x) - 4(l - x)^3\}.$$

26. *Beams with Fixed Ends*

The deflection line can here be assumed as a trigonometric series of the form

$$y = \sum_{n=1}^{\infty} \frac{1}{2} a_n \left(1 - \cos\frac{2n\pi x}{l}\right),$$

every term of which satisfies the end conditions ($y = 0$, $y' = 0$) at both ends of the beam.

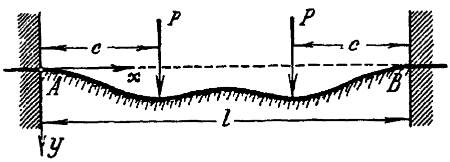

FIG. 69

Since every term of this series is symmetrical with respect to the center line ($x = l/2$) of the beam, formulas can be developed from it only for symmetrical loadings.

If we apply the strain-energy method in the same manner as was done in the previous types of beams, the assumption above will lead to the deflection formulas below.

a. Two Symmetrical Concentrated Forces (Fig. 69)

$$y = \frac{Pl^3}{4\pi^4 EI}\sum_{n=1}^{\infty}\frac{\left(1 - \cos\frac{2n\pi c}{l}\right)\left(1 - \cos\frac{2n\pi x}{l}\right)}{n^4 + \frac{3}{16}\frac{kl^4}{\pi^4 EI}}. \tag{71}$$

b. Two Symmetrical Concentrated Moments (Fig. 70)

$$y = \frac{Ml^2}{2\pi^3 EI}\sum_{n=1}^{\infty}\frac{n\sin\frac{2n\pi c}{l}\left(1 - \cos\frac{2n\pi x}{l}\right)}{n^4 + \frac{3}{16}\frac{kl^4}{\pi^4 EI}}. \tag{72}$$

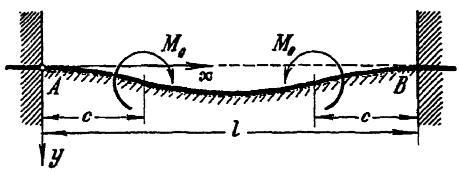

FIG. 70

c. Symmetrically Distributed Loading (Fig. 71)

$$y = \frac{ql^3}{8\pi^4 EI} \sum_{n=1}^{\infty} \frac{\left(l - 2c + \frac{l}{n\pi} \sin \frac{2n\pi c}{l}\right)\left(1 - \cos \frac{2n\pi x}{l}\right)}{n^4 + \frac{3}{16} \frac{kl^4}{\pi^4 EI}}. \tag{73}$$

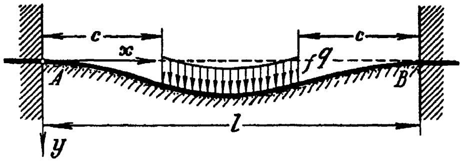

FIG. 71

III. *Applications*

27. *Examples*

1. In Figure 72 is shown the cross section of an aqueduct. In the direction perpendicular to the plane of the paper this section is assumed to continue in the same form; thus we can consider the object as a two-dimensional problem. The question is to find the pressure distribution in the subsoil and the moment diagram for the bottom plate. The modulus of elasticity of the structure (concrete) is $E = 2.5 \times 10^6$ lbs./in.^{2}; the modulus of foundation, $k_0 = 165$ lbs./in.^{3} The bottom plate can be regarded as a beam on elastic foundation, with a uniformly distributed loading (the weight of the bottom plate itself and the water pressure along its whole length) and subjected in addition at its ends to concentrated forces (weight of the side walls) and moments (due to the hydraulic pressure on the side walls).

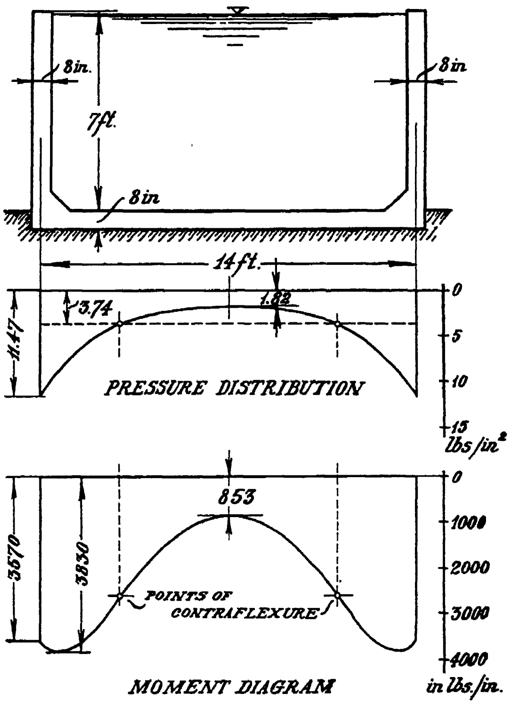

FIG. 72

Computing these loadings for a section of unit width, 1 in., of the structure, we have:
Weight of the bottom plate:

$$8 \text{ in.} \times 0.0875 \text{ lbs./in.}^3 = 0.70 \text{ lbs./in.}^2;$$

Weight of the water:

$$84 \text{ in.} \times 0.0361 \text{ lbs./in.}^3 = 3.04 \text{ lbs./in.}^2, \qquad q = 3.74 \text{ lbs./in.}^2;$$

Weight of the wall:

$$P = 86 \text{ in.} \times 8 \text{ in.} \times 0.0875 \text{ lbs./in.}^3 = 60 \text{ lbs./in.};$$

Moment of the hydraulic pressure:

$$M = \tfrac{1}{6} \times 84^3 \times 0.0361 = 3570 \text{ in. lbs./in.}$$

Hence our problem is to find the deflection and moment curve of a beam, on elastic foundation, subjected to the loading shown in Figure 73. The distributed loading will cause only a uniform compression in the foundation, of the value $p = 3.74$ lbs./in.^{2} The deformations and moments produced by the P and M end loadings can be computed from (37 a and c) and (38 a and c) respectively. The result of this computation is shown in Figure 72. The maximum pressure is at the ends: $p_{max} = 11.47$ lbs./in.^{2}, while the maximum value of the bending moment, $M_{max} = -3830$ in. lbs./in., occurs at some distance from the ends, as shown in Figure 72.

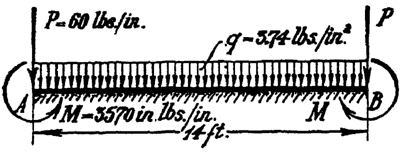

FIG. 73

In connection with this problem we shall now illustrate the previously mentioned applicability of the series method to design problems. Let us suppose that the p_{max} soil pressure obtained above is greater than the pressure permissible for the subsoil under consideration and let us set the problem of finding that value I for the bottom plate which will secure a more uniform pressure distribution under the structure and will lower the value of p_{max} to 8 lbs./in.^{2} Subtracting from the given 8 lbs./in.^{2} the uniform loading of the intensity $q = 3.74$ lbs./in.^{2} (Fig. 73), we obtain $p = y_0k = 4.26$ lbs./in.^{2} as the maximum permissible pressure due to the concentrated loadings at the ends.

Using (60) and (61), we can write the requirement above as

$$y_0k = \frac{2P}{l} + \frac{2P\dfrac{4}{\pi^2} + 2M\dfrac{2}{l}}{\dfrac{\pi^4E}{2l^3k}I + \dfrac{l}{2} - \dfrac{4l}{\pi^2}} = 4.26 \text{ lbs./in.}^2$$

Substituting the given numerical values in this equation and expressing I as unknown, we have the answer that the moment of inertia, satisfying our condition, must be $I = 140$ in.^{4}, which corresponds to a thickness $t = 11.9$ in. for the bottom plate instead of the 8-in. thickness used in the example. Checking this conclusion by exact formulas derived from (37a) and (38a), corresponding to $I = 140$ in.^{4}, we obtain the pressure at the ends due to the end loadings as $p = 4.46$ lbs./in.^{2}, and adding to this the uniformly distributed 3.74 lbs./in.^{2} loading, we get the maximum pressure as $p_{max} = 8.20$ lbs./in.^{2}, which exceeds by only

2.5 per cent the required limit. Though in this example we had $\lambda l = 4.20$, which is beyond the range of bars of medium length, the results obtained by the series method are still accurate enough for practical purposes.

2. Consider a thin-walled cylindrical tube subjected to internal pressure p and supplied at equal intervals l with rigid reinforcing rings, as shown in Figure 74. In § 12 (p. 35) we have already analyzed the problem when only one rigid ring was applied to the tube. There we found that, while in a plane tube under internal pressure p the maximum stress is in the circumferential direction and its value is $\sigma_c = pR/t$, t denoting the wall thickness and R the middle radius of the tube, upon application of the ring a bending stress is produced in the longitudinal fibers of the beam, and this stress is $\sigma_m = 1.82\, pR/t$. The problem

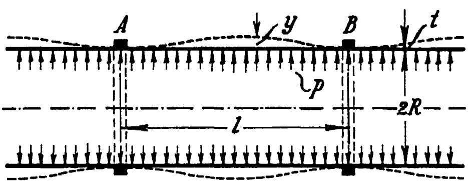

FIG. 74

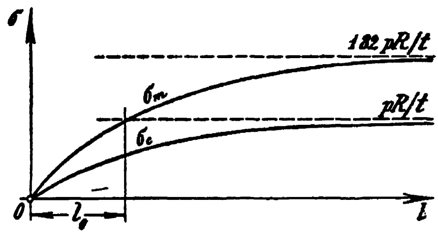

FIG. 75

we shall investigate now will be the effect of the spacing of the rings upon the maximum bending stress in the tube.

If a series of rigid rings is applied at equal l intervals to the tube and the spacing is increased from zero to very large values of l, we find that the σ_c and σ_m stresses must increase from zero values to $\sigma_c = pR/t$ and $\sigma_m = 1.82\, pR/t$ respectively in the manner indicated in Figure 75. Hence we see that there must be a certain spacing l_0 where the maximum bending stress produced by the rings will be just equal to the circumferential stress value in a plane tube under internal pressure. This l_0 spacing of the rings can be determined in the following way.

A longitudinal element of the tube between two reinforcing rings will be regarded as a beam fixed at the two ends, subjected to a uniformly distributed loading p and supported, at the same time, along its entire length on an elastic foundation. The modulus of this foundation will be, according to (25), $k = Et/R^2$ and the moment of inertia of the beam $I = t^3/12(1 - \mu^2)$, with t denoting the wall thickness and R the middle radius of the tube. By means of (50c) the condition $\sigma_m = \sigma_c$ for l_0 can be written as

$$\frac{6M_A}{t^2} = \frac{6}{t^2}\frac{p}{2\lambda^2}\frac{\text{Sinh}\,\lambda l_0 - \sin\lambda l_0}{\text{Sinh}\,\lambda l_0 + \sin\lambda l_0} = \frac{pR}{t},$$

where, with $\mu = 0.3$,

$$\lambda = \sqrt[4]{3(1-\mu^2)}\cdot\frac{1}{\sqrt{Rt}} = 1.285\,\frac{1}{\sqrt{Rt}}$$

Introducing this expression in the equation above we have the condition in the simplified form:

$$\frac{\text{Sinh } \lambda l_0 - \sin \lambda l_0}{\text{Sinh } \lambda l_0 + \sin \lambda l_0} = \sqrt{\frac{1 - \mu^2}{3}} = 0.5505.$$

By the trial-and-error procedure we find this equality is satisfied by a value of $\lambda l_0 = 1.899$, from which we have the required spacing

$$l_0 = \frac{1.899}{1.285} \sqrt{Rt} = 1.478 \sqrt{Rt}.$$

If the spacing is larger than l_0 given by this equation, then the application of the rings will increase the maximum stress value in the tube. A decrease of stress will occur only if $l < l_0$. It is seen from the formula above that, compared to the radius of the tube, l_0 will have a rather small value.

3. Though the relationship between stresses and deformations in earth foundations is, in general, of a very complex nature, not infrequently we find subsoils which follow the simple law assumed for our elastic foundation (p. 2) accurately enough to permit a mathematical analysis to be developed on this basis. If this is the case, stresses and deflections occurring in sheet pilings and similar structural elements can be readily calculated by the formulas given in the first part of this chapter. Sheet pilings, for instance, may be regarded as cantilever beams partly surrounded by an elastic foundation, as shown in Figure 76. Under the action of the loading, which is usually applied at the unsupported part of the cantilever, the beam will deform elastically and will assume a position in which the active and passive earth pressures arising from the displacements in the subsoil maintain equilibrium with the external loading.

According to the Rankine theory, if there is no cohesion between the elementary particles of the soil, an active earth pressure of the amount $p_1 = \gamma x \cdot (1 - \sin \varphi)/(1 + \sin \varphi)$ is present at a depth x under the surface; the resistance of the soil against displacement, that is, the passive earth pressure at the same depth, can be expressed as $p_2 = \gamma x \cdot (1 + \sin \varphi)/(1 - \sin \varphi)$, γ denoting the specific weight of the earth under consideration and φ representing its angle of friction. The value $p_2 - p_1 = \gamma x \cdot 4 \sin \varphi/(1 - \sin^2 \varphi)$ defines the maximum pressure which can be resisted by the soil and determines the limit of its static equilibrium. This point should be taken into consideration in the analysis of sheet pilings and similar constructions. If the modulus of foundation k is assumed in such cases to increase proportionally with the depth x, the pressure distribution along the beam will be defined by $p = kxy$, where y denotes the transverse displacement of the beam. This pressure p must stay at every point within the maximum resistance, $p_2 - p_1$, of the soil.

The deflection curve of the types of beams under consideration will be such that its maximum ordinate will occur right at the surface of the earth, where, to a certain depth x_0, the pressure $p = kxy$ will necessarily exceed the resistance

$p_2 - p_1$. For this reason, along the length x_0, where the static equilibrium is overcome, the resistance of the earth will not be considered, and the elastic foundation will be taken into account from point A on, at which point we have $p = p_2 - p_1$.

The case when the modulus of foundation varies proportionally with the depth, which is compatible with the Rankine theory of earth pressure, will be discussed in the next chapter. At this place only the method of solution when the modulus of foundation is constant will be illustrated.

The portion A–B of the beam (Fig. 76) with a length $l = L - x_0$, involving the unknown depth x_0, is subjected at end A to a concentrated force P and a bending moment $M_0 = P(h + x_0)$. The requirement that at point A

$$ky_A = \frac{4 \sin \varphi}{1 - \sin^2 \varphi} \gamma x_0$$

can be written from (39a) and (40a) as

$$\frac{2P\lambda}{\text{Sinh}^2 \lambda l - \sin^2 \lambda l} [\text{Sinh } \lambda l \text{ Cosh } \lambda l - \sin \lambda l \cos \lambda l$$
$$+ \lambda (h + x_0)(\text{Sinh}^2 \lambda l + \sin^2 \lambda l)] = \frac{4 \sin \varphi}{1 - \sin^2 \varphi} \gamma x_0.$$

From this equation, by a trial-and-error procedure, the unknown x_0 can be determined. Having it, we can obtain diagrams of deflection, moment, and so on for the A–B portion by using (39 a–d) and (40 a–d). As a numerical example, let us assume the following values: $L = 72$ in.; $h = 36$ in.; $E = 2 \times 10^6$ lbs./in.^{2}; $I = 140$ in.^{4}; $k = 180$ lbs./in.^{2}; $\gamma = 0.10$ lbs./in.^{3}; and $\varphi = 25^0$. Hence we have $\lambda = \sqrt[4]{k/4EI} = 0.020$ in.$^{-1}$ By trial and error we find that $x_0 = 19.5$ and $l = L - x_0 = 52.5$ satisfy the equation above. Having the unknown x_0 and having computed the deflection and pressure distribution along the portion A–B from (39a)–(40a), we get the result shown in Figure 76.

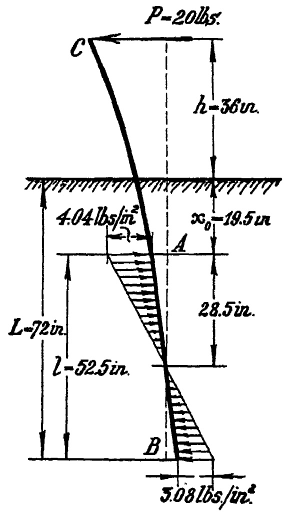

FIG. 76

4. Consider an arrangement in which two main longitudinal beams are supported by closely laid transverse bars and the whole structure is placed on an elastic foundation (Fig. 78). The longitudinal beams are assumed to be infinitely long, and the problem is to determine the deflection surface of this infinite strip under a concentrated load P acting at point A on one of the main longitudinal beams.

The dimensions of the construction are shown in Figure 78. The transverse bars are assumed to be 1 in. wide. We shall take for these bars $E = 5.0 \times 10^6$ lbs./in.^{2}, and for the foundation, $k_0 = 240$ lbs./in.^{3} The first step will be to determine the deflection of one transverse bar under symmetrically and anti-symmetrically applied concentrated forces $P = 1000$ lbs. The values of the deflection ordinates were computed by (37a) and (39a), and the results are shown in Figure 77. Since in our structure the main girders are supported at the ends of the transverse bars, the end deflections shown in Figure 77 will define a modulus of foundation for the main girders in symmetrical and antisymmetrical loadings.

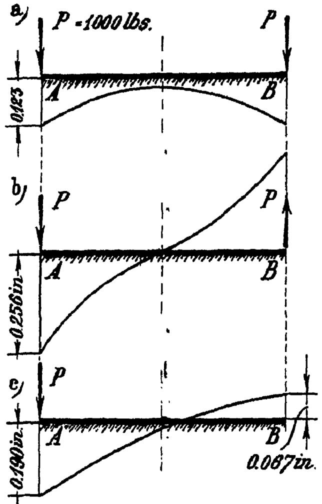

FIG. 77

Let us first consider the situation when the two main beams are subjected to equal and downward-acting forces $P/2$ at points A and B (case I). In this symmetrical case the modulus k for each of the main beams will be, from Figure 77a,

$$k_I = \frac{1000 \text{ lbs.}}{0.123 \text{ in.}} \text{ per inch} = 8130 \text{ lbs./in.}^2,$$

which, if we take $E = 1.25 \times 10^6$ lbs./in.^{2} for the modulus of elasticity of the main beams, gives

$$\lambda_I = \sqrt[4]{\frac{8130}{4 \times 1.25 \times 10^6 \times \frac{4}{3}(20)^3}} = 0.0198 \text{ in.}^{-1}$$

In the corresponding antisymmetrical loading, where there is a downward-acting force $P/2$ at A and an upward-acting $P/2$ at B (case II), the deflection of each main girder can evidently be computed, from Figure 77b, with

$$k_{II} = \frac{1000 \text{ lbs.}}{0.256 \text{ in.}} \text{ per in.} = 3900 \text{ lbs./in.}^2$$

and

$$\lambda_{II} = \sqrt[4]{\frac{3900}{4 \times 1.25 \times 10^6 \times \frac{4}{3}(20)^3}} = 0.0164 \text{ in.}^{-1}$$

Computing the deflection of each of the main beams, first with the k_I, λ_I and then with the k_{II}, λ_{II} values, and superposing cases I and II, we obtain the deflection surface in Figure 78 as the answer to the original problem. In each case the deflection line of the transverse bars will be similar, proportional to the end ordinates, and in every instance points of zero deflection will lie on a straight line parallel with the main girders. Superposing cases I and II, however, we

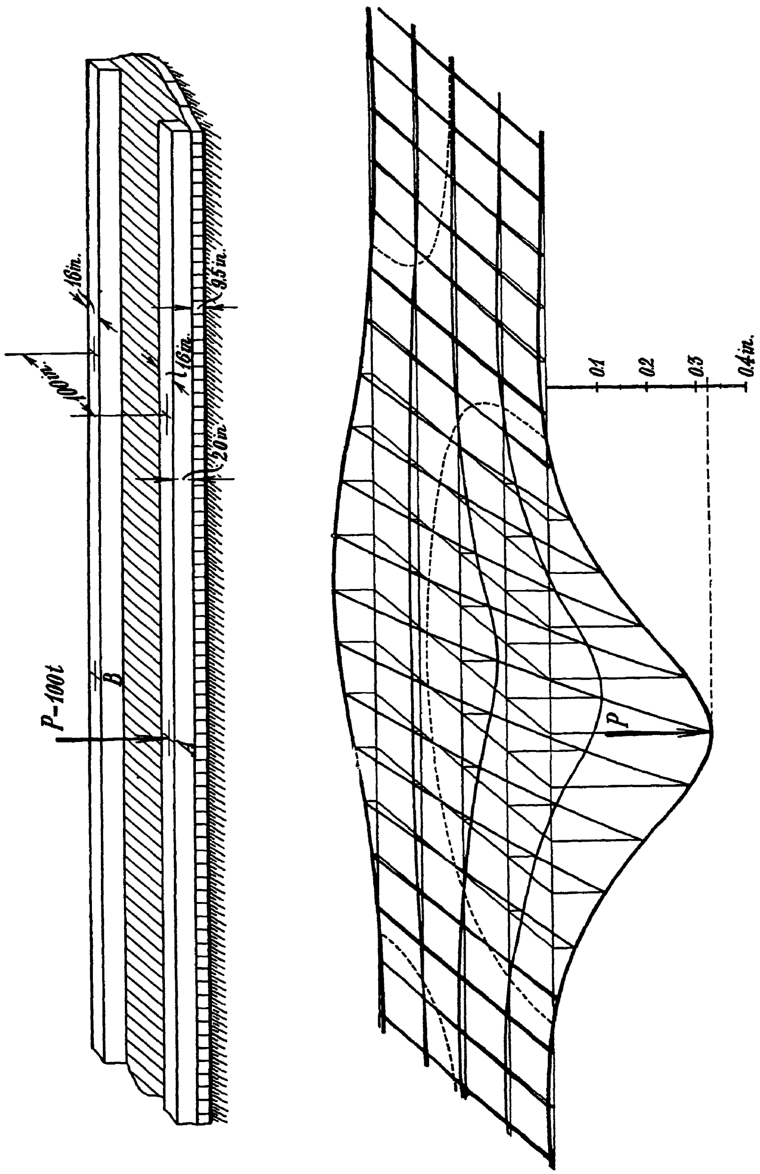

FIG. 78

obtain a three-dimensional solution. In Figure 78 the dotted lines connecting points of zero deflection illustrate the three-dimensional character of the problem.

5. It has been shown in Chapter II, page 30, that a longitudinal element of a cylindrical shell under axially symmetrical loading can be regarded as a beam on elastic foundation, in which event the modulus of the foundation k and the characteristic λ are

$$k = \frac{Et}{R^2} \text{ and } \lambda = \sqrt[4]{3(1 - \mu^2)}\ \frac{1}{\sqrt{Rt}},$$

where R is the middle radius, t the thickness of the cylinder, E the modulus of elasticity, and μ Poisson's ratio for the material.

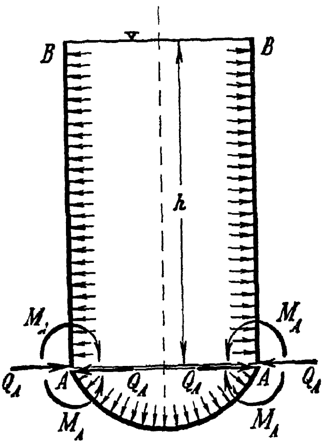

FIG. 79

On this basis many stress problems in connection with cylindrical containers and pressure vessels can be analyzed. In the general case, when a cylindrical body is joined to a flexible drumhead (Fig. 79), the calculation can be carried out in the same manner as for statically indeterminate structures. The container will first be pictured as cut up into its two main structural parts, the cylindrical body and the head, and it will be assumed that each of these parts can freely deform under the action of the loading, which usually is an inside pressure on the container walls. As a result of these deformations we obtain along A–A a discontinuity in displacements and rotations between the neighboring ends of the cylindrical body and the head. In order to maintain the continuity of the material (continuous displacement and slope) in this region, we shall have to apply shearing forces Q_A and bending moments M_A, uniformly distributed along the circle of juncture A–A. The values of M_A and Q_A can be determined from the two simultaneous continuity requirements mentioned above.*

To be able to determine these unknowns, M_A and Q_A, we must know the deformation of the cylindrical body and the head due to the internal pressure as well as that due to the unit values of the M_A and Q_A quantities respectively. Stresses and deformations attributable to internal pressure can be determined for each part separately by simple considerations of statics, which, since the bending resistance of the elements is usually neglected in such cases, furnish the so-called membrane stresses and deformations. Let us assume that this mem-

* In addition to the loading M_A and Q_A shown, there will also be a distributed axial force around circle A–A due to the reaction of the head, but the effect of this axial force on the displacements due to M_A and Q_A in the cylinder is quite negligible in most engineering structures, as is shown in Chapter VI, p. 138.

brane analysis gives a radial displacement y_c and a rotation θ_c for the cylindrical part and y_h and θ_h values for the head along the circle A–A. We shall then have at this place a discontinuity $y_c - y_h$ in displacement and $\theta_c - \theta_h$ in slope.

Displacements and rotations of the end A of the cylinder due to unit values of Q_A and M_A can be determined from (39 a and b) and (40 a and b) and will be denoted by y_c^Q, θ_c^Q, and y_c^M, θ_c^M respectively. Corresponding values for the spherical head, y_h^Q, θ_h^Q and y_h^M, θ_h^M, can be computed from equation (150 a–c), in Chapter IX. If we are dealing with a flat plate head these displacement values can be obtained from any book concerned with the theory of thin plates. Having these data, we can write the continuity in deflection and slope along the A–A circle as

$$(y_c^Q + y_h^Q)Q_A + (y_c^M + y_h^M)M_A = y_c - y_h \,,$$

$$(\theta_c^Q + \theta_h^Q)Q_A + (\theta_c^M + \theta_h^M)M_A = \theta_c - \theta_h \,,$$

from which the unknown Q_A and M_A quantities can be determined. The stresses set up by Q_A and M_A are usually called *discontinuity stresses*, and they have primary importance in the design and strength of containers and pressure vessels.

In the general case, when we deal with flexible container heads, the analysis must be carried out according to the scheme above. The computation will be greatly simplified, however, if the construction of the head is so rigid that its deformation, in comparison with that of the cylinder, becomes negligibly small. In such a case an axial strip of the cylindrical portion A–B in Figure 79 can be regarded as a cantilever beam fixed at the end A, supported on an elastic foundation and loaded by a uniform or a triangular (hydraulic) distributed loading, according to the nature of the internal pressure in the container. In either case of loading the solution can be readily obtained from the formulas developed for cantilever beams in § 22.

As an example, let us consider the cylindrical tank in Figure 80, with the dimensions: $R = 360$ in.; $t = 14$ in.; $h = 312$ in.; material: reinforced concrete $E = 4.25 \times 10^6$ lbs./in.^{2}; $\mu = 0.25$; tank filled with water, specific gravity $\gamma = 0.0361$ lbs./in.^{3} The bottom plate will be assumed to be perfectly rigid, and thus an element A–B can be regarded as a cantilever beam subjected to a triangular distributed load with a maximum value at A of $q_0 = \gamma h$. We have

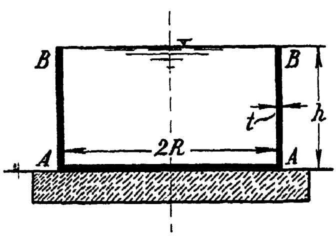

FIG. 80

$$k = \frac{Et}{R^2} = \frac{4.25 \times 10^6 \times 14}{(360)^2} = 458 \text{ lbs./in.}^2$$

and

$$\lambda = \sqrt[4]{3(1-\mu^2)}\,\frac{1}{\sqrt{Rt}} = \sqrt[4]{3 \times 0.9375}\,\frac{1}{\sqrt{360 \times 14}} = 0.0182 \text{ in.}^{-1}$$

Since $\lambda h = 0.0182 \times 312 = 5.68$ is sufficiently large, accurate results can be obtained by the simplified formulas in (54 a–d). Putting $l = h$ and $q_0 = \gamma h$, we have for the unknown M_A and Q_A quantities

$$M_A = -\frac{\gamma}{2\lambda^3}(\lambda h - 1) = 14{,}000 \text{ in. lbs./in.},$$

$$Q_A = \frac{\gamma}{2\lambda^2}(2\lambda h - 1) = 564 \text{ lbs./in.}$$

The distribution of bending moments and shearing forces along A–B can be computed from (54 c and d) respectively. The circumferential hoop force N

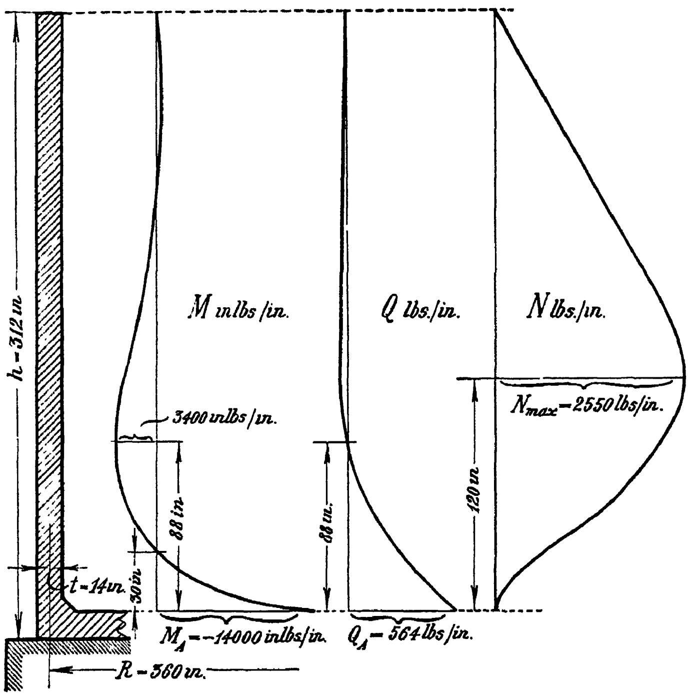

FIG. 81

lbs./in. can be obtained from the deflection formula (54a) as $N = -kyR$. The results of these computations are shown in Figure 81. It should be noted that in calculating the maximum stress in the circumferential direction we must add, according to (27), to the circumferential normal stress the effect of circumferential bending $\sigma_c = N/t + 6\mu M/t^2$.

6. Figure 82 shows schematically the construction of a commutator for an electrical machine. The copper commutator bars with mica layers between them are held together by the radial forces which result as the bolts are tightened up on the V rings shown in the figure. Because of this manner of assembly

such a commutator construction is referred to as the *arch-bound* type. Since the overhanging parts A–B of the bars are portions of a cylindrical shell, they can be computed as beams on an elastic foundation. In the calculation it is to be assumed that the commutator is built up of two materials, copper and mica; consequently, the modulus of foundation $k = Et/R^2$ used when calculating cylindrical tubes will need some modification in this case.

Denoting by E_c and E_m the modulus of elasticity of copper and mica, and by b_c and b_m the width of the copper bars and mica layers respectively, we get

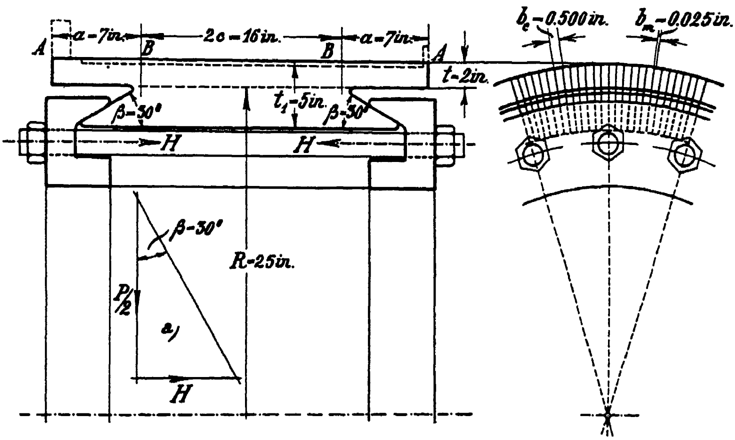

FIG. 82

the unit compression produced by a circumferential pressure p between the bars as

$$\frac{p}{E_c}\frac{b_c}{b_c + b_m} + \frac{p}{E_m}\frac{b_m}{b_c + b_m} = \frac{p}{E_c}\frac{b_c + \dfrac{E_c}{E_m}b_m}{b_c + b_m}.$$

Hence the compressibility of the composite tube will be the same as that of a homogeneous tube with an elastic modulus:

$$E = E_c\frac{b_c + b_m}{b_c + \dfrac{E_c}{E_m}b_m}.$$

This is the value of E which will have to be used in the formula for the modulus k. Taking the dimensions $b_c = 0.500$ in. and $b_m = 0.025$ in. from Figure 82 and assuming* $E_c/E_m = 3$, we have $E = 0.912\ E_c$. The same effect is considered

* The use of the actual, nonlinear, stress-strain relation in mica would have only a negligible effect in this calculation.

when computing the moment of inertia of the A–B beams, by taking for unit width

$$I = \frac{b_c}{b_c + b_m}\frac{1 \times t^3}{12} = 0.952\frac{t^3}{12}.$$

With these values we have

$$\lambda = \sqrt[4]{\frac{k}{4E_cI}} = \sqrt[4]{\frac{3 \times 0.912}{0.952}}\frac{1}{\sqrt{Rt}} = 1.300\frac{1}{\sqrt{Rt}}.$$

Taking $R = 25$ in., $t = 2$ in., and $a = 7$ in., we get

$$\lambda = 0.184 \text{ in.}^{-1} \quad \text{and} \quad \lambda a = 1.288.$$

The middle part B–B of the commutator bar can be assumed to be perfectly rigid, and thus the overhanging parts A–B can be looked upon as cantilever beams fixed against rotation at the base B. In the stress calculation two main types of loading will be considered: the assembly forces and the inertia forces due to rotation.

Denoting by H the assembly force produced by the bolts per 1-in. run of the circumference, if friction is not considered at the contact area under the V ring, we see that a unit width of the bars will be subjected to a radial compressive force $P = 2H \cot \beta$ (Fig. 82). Let us assume that because of this centrally applied force P the rigid middle part B–B will undergo a uniform radial displacement y_B. This will correspond to a uniformly distributed reaction $2c\alpha k y_B$ along the length B–B, where the factor

$$\alpha = \frac{t_1}{t} + \frac{\sqrt{3}(t_1 - t)^2}{2ct}$$

takes into account the increase of area along B–B in comparison with the uniform area along the beams A–B. The rest of the reaction,

$$P - 2c\alpha k y_B = 2Q_B,$$

will be transmitted in the form of a shearing force Q_B to each of the two flexible beams A–B. The deflection of these beams at points B must equal the value y_B, which condition can be written from (41a) as

$$y_B = \frac{2Q_B\lambda}{k}\frac{\text{Cosh}^2 \lambda a + \cos^2 \lambda a}{\text{Sinh } 2\lambda a + \sin 2\lambda a}.$$

Substituting here the expression above for Q_B, we have the deflection y_B as a function of the loading P:

$$y_B = \frac{\lambda P}{k}\frac{1}{2c\alpha\lambda + \dfrac{\text{Sinh } 2\lambda a + \sin 2\lambda a}{\text{Cosh}^2 \lambda a + \cos^2 \lambda a}}. \tag{c}$$

Taking $H = 5000$ lbs. per inch circumference, we have $P = 17{,}300$ lbs./in., and, since $t = 2$ in. and $t_1 = 5$ in., we get $\alpha = 2.987$. Using these values and $2c = 16$, $\lambda = 0.184$, $\lambda a = 1.288$, we have from the formulas above

$$y_B = 0.094\frac{P\lambda}{k} \quad \text{and} \quad Q_B = \tfrac{1}{2}P(1 - 0.826) = 0.087P = 1500 \text{ lb./in.}$$

The bending moment in sections B can be calculated from (41c) as $M = 4310$ in. lbs./in., which then gives as the maximum bending stress produced by the assembly forces

$$\sigma_a = \frac{4310}{0.952\frac{t^2}{6}} = 6790 \text{ lbs./in.}^2$$

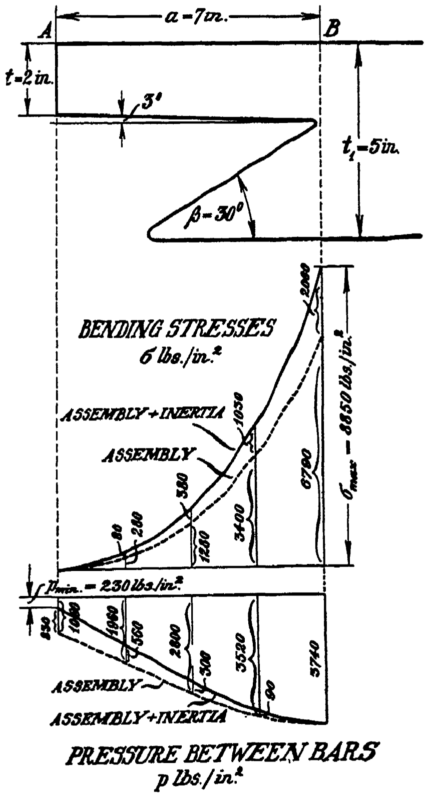

FIG. 83

The circumferential pressure $p = kyR/t$ is maximum at B (and the same over the rigid center portion), where its value is

$$p_B = 0.094P\lambda R/t = 3740 \text{ lbs./in.}^2$$

At the end A, since from (41a), $y_B/y_A = 3.530$, the circumferential pressure reduces to $p_A = 1060$ lbs./in.^{2}

Stresses due to inertia forces can be computed by assuming the portion A–B as a cantilever beam on an elastic foundation, fixed at the end B and loaded with a uniformly distributed loading

$$q = \frac{\gamma t}{g}\omega^2 R,$$

where γ is the specific weight of the bars, g the gravitational acceleration, and ω the angular velocity of the commutator.

Taking the speed of rotation equal to 450 r.p.m., we have $\omega = 47.12$ sec.$^{-1}$ Then putting $\gamma = 0.320$ lbs./in.^{3} and $g = 386$ in./sec.^{2}, we have $q = 92.1$ lbs./in. per inch. The bending moment in section B calculated from (52c) will be $M = 1310$ in. lbs./in., and will produce a bending stress

$$\sigma_c = \frac{1310}{0.952\frac{t^2}{6}} = 2060 \text{ lbs./in.}^2$$

The circumferential pressure, or pressure relief in this case, produced at point A by the centrifugal forces can be calculated from (52a); taking $p_A = (kR/t)y_A$, we then get

$$y_A = 0.719 \frac{q}{k} \quad \text{and} \quad p_A = 0.719 \frac{qR}{t} = 830 \text{ lbs./in.}^2$$

Adding up the centrifugal and assembly stresses, we obtain the maximum bending stress at B as

$$\sigma = \sigma_a + \sigma_c = 6790 + 2060 = 8850 \text{ lbs./in.}^2$$

The circumferential pressure between the bars will be constant $p = 3740$ lbs./in.^{2} along the rigid center portion B–B. Along beams A–B the centrifugal loading will cause a release in p, and we shall have its minimum value at A as

$$p_A = 1060 - 830 = 230 \text{ lbs./in.}^2$$

The distribution of bending stresses σ and arching pressures p along the beams A-B is shown in Figure 83. These curves have been calculated from (41c), (52c) and (41a), (52a), respectively.

7. Let us assume that in coal mining a gangway is driven into a seam of coal which is supporting a rock layer of limited thickness, as shown in Figure 84. By driving the gangway the originally uniform stress distribution is disturbed, for in the coal layer close to the gangway an increase of pressure is produced, and in the rock layer, which can be regarded as a beam in this case, bending stresses are set up.

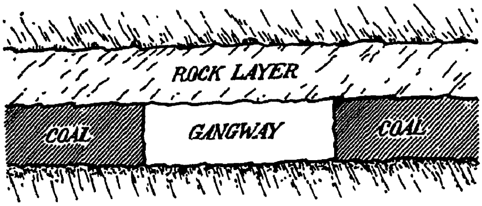

FIG. 84

As a rule, the modulus of elasticity for coal (E_c) and for rock species (E_r) can be definitely established in any particular situation which makes feasible the stress calculation in the elastic range. Since E_r is usually much larger than E_c, we can disregard in an approximate analysis the continuity in the coal layer. This assumption then reduces the problem to the simple case of a beam partially supported on an elastic foundation of the $p = ky$ type and loaded with a uniformly distributed loading, the solution of which is given in § 23 (p. 68).

The same problem has been investigated experimentally by E. Lehr.* In his experiments the upper layer of rocks was replaced by a hard rubber bar laid on various foundations of hard and soft rubber representing the sublayer of coal. The bar was loaded with a uniformly distributed loading and the deflection

* "Modellversuche an Balken auf elastischer Unterlage zur Klärung der Spannungsverteilung im Hangenden von Abbauörten," Forschungsheft 372, Beilage zur *Forschung auf dem Gebiete des Ingenieurwesens*, Ausgabe B, Band 6 (May–June, 1935), pp. 22–33.

curve was measured, from which, by a method of graphical differentiation, the bending stresses were derived. Since in Lehr's publication no attempt was made at an analytical solution of the problem, such a solution will be demonstrated here by using the dimensions from one of his examples. In order to be able to compare the computation with the experimental results, we shall employ the kilogram-meter system.

The dimensions of the experimental setup shown in Figure 85 are: $l = 15$ cm.; $L = 20$ cm.; $k_0 = 30$ kg./cm.^{3}; $k = bk_0 = 60$ kg./cm.^{2}; $E = 23{,}000$ kg./cm.^{2}; $I = \frac{4}{3}$ cm.^{4} Hence we have $\lambda = \sqrt[4]{k/4EI} = 0.149$ cm.$^{-1}$, and $\lambda l = 2.235$,

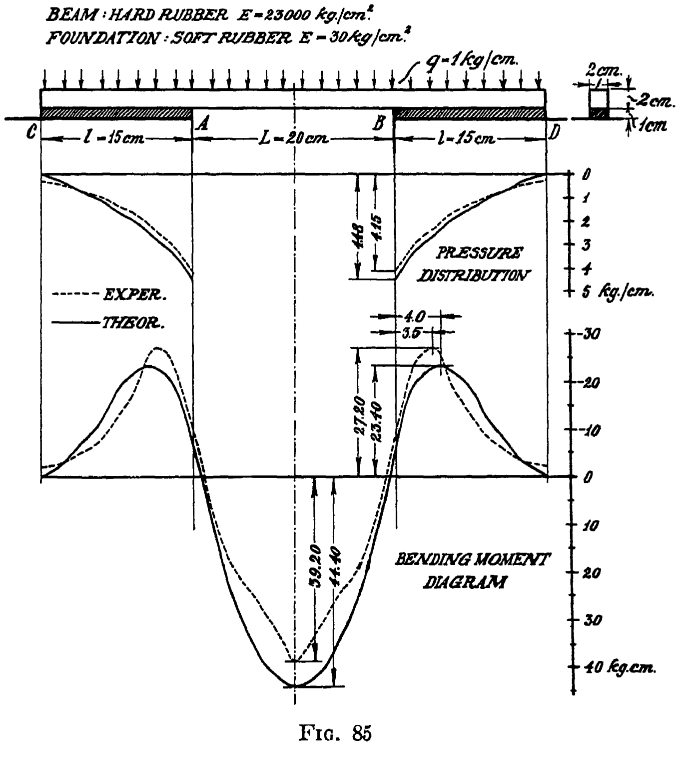

FIG. 85

$\lambda L = 2.980$. The distributed loading $q = 1$ kg./cm. above the side portions AC and BD will cause only uniform compression and displacement. The loading above the free span, on the other hand, will produce a bending of the beam. Using the given dimensions, we have from equation (b) on page 68

$$M_A = M_B = -5.60 \text{ kg./cm.} \qquad \text{and} \qquad Q_A = -Q_B = 10.0 \text{ kg.}$$

Putting these values into (39 a–c) and (40 a–c) we obtain the deflection (pressure) and bending-moment diagrams which are shown in Figure 85, in comparison with the experimental results. The high local curvatures in the experimental bending-moment diagram and the presence of bending ordinates at the free ends C and D seem to show experimental errors. In view of these errors and of the fact that in this case there was a definite material continuity in the foundation which our theory, in turn, did not take into account, there is a remarkable agreement between the analytical and experimental results.

If the side spans AC and BD are considered infinitely long, which more closely approximates the actual situation in the mine, the computation takes a still simpler form, and then (56 a–d) can be used.

Ludwig Prandtl* has suggested the theoretical analysis of an experimental setup of this same arrangement to investigate the failure of brittle materials in tension. If we merely reverse the direction of the distributed loading in Figure 85, so that it now represents tensile forces, and imagine the A–B span to be an initial crack in the material, we have a situation similar to what may exist in the inside of a brittle material in tension. The maximum tensile stress will then occur at the A and B end points of the crack and its intensity will be increased as the A–B distance increases. Prandtl, reasoning in this way, explains several phenomena (why longer cracks spread more easily, and so on) observed in experiments with brittle materials.

* "Ein Gedankenmodell für den Zerreissvorgang spröder Körper," *Zeitschrift für angewandte Mathematik und Mechanik*, Band 13 (1933), Heft 2, pp. 129–133.

CHAPTER V

BEAMS OF VARIABLE FLEXURAL RIGIDITY AND VARIABLE MODULUS OF FOUNDATION

28. *Variation in Steps*

A beam whose flexural rigidity varies in steps can be regarded as composed of shorter prismatic beams, formulas for which have been developed in the previous chapters. Thus the problem can be analyzed by the usual method of computing statically indeterminate structures.

Imagine that the beam, which is subjected to the given loading, is cut through at the places where there is an abrupt variation in the cross section (Fig. 86). This cutting produces separate prismatic bars, the deflections of which we can determine from the formulas previously developed. At every point of separation there will be, because of the loading, a relative displacement and slope difference between the ends of the neighboring bars. Denote these quantities at the kth point of separation by the symbols δ_{0k} and θ_{0k}. On the original beam there is continuity at that point, and thus no sharp change of deflection or slope between the neighboring sections can occur; instead of these displacement quantities, moments and shearing forces act to bring about and maintain the continuity of the bar.

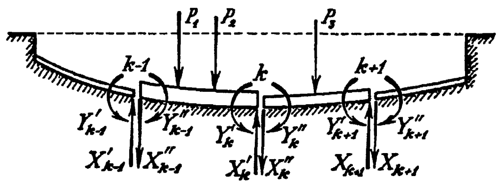

FIG. 86

Denoting these shearing forces and moments acting at the kth point* by X'_k, Y'_k and X''_k, Y''_k, according to whether they act on the left or on the right ends meeting at the point k, and denoting the deflection and angular displacement due to unit values of each of these forces by $\delta^{X'}_{k,k-1}$, $\theta^{X''}_{k,k-1}$, and so on, in which the first subscript shows where the force is acting and the second shows where the deflection is produced, we can write the equations of continuity for the kth point as

$$\left.\begin{aligned} &X''_{k-1}\delta^{X''}_{k-1,k} + Y''_{k-1}\delta^{Y''}_{k-1,k} + X_k(\delta^{X'}_{k,k} + \delta^{X''}_{k,k}) + Y_k(\delta^{Y'}_{k,k} + \delta^{Y''}_{k,k}) \\ &\qquad\qquad + X'_{k+1}\delta^{X'}_{k+1,k} + Y'_{k+1}\delta^{Y'}_{k+1,k} + \delta_{0k} = 0, \\ &X''_{k-1}\theta^{X''}_{k-1,k} + Y''_{k-1}\theta^{Y''}_{k-1,k} + X_k(\theta^{X'}_{k,k} + \theta^{X''}_{k,k}) + Y_k(\theta^{Y'}_{k,k} + \theta^{Y''}_{k,k}) \\ &\qquad\qquad + X'_{k+1}\theta^{X'}_{k+1,k} + Y'_{k+1}\theta^{Y'}_{k+1,k} + \theta_{0k} = 0. \end{aligned}\right\} \quad (74)$$

* These forces differ only in sign: $|X_k'| = |X_k''| = X_k$ and $|Y_k'| = |Y_k''| = Y_k$, as shown in Figure 86.

Forces acting at the $k - 2$ or $k + 2$ points have no effect on the displacements occurring at point k. For each point where there is a step in the cross section we can write two equations of continuity of the same sort as those in (74); accordingly there will be two unknown quantities (X and Y) to determine at each of these points. These equations are of the same type as the well-known three-moment equations. The scheme of the equation system so obtained will have a diagonal symmetry, since $\delta^{X''}_{k-1,k} = \delta^{X'}_{k,k-1}$; furthermore, $\theta^{X}_{k-1,k} = \theta^{Y}_{k,k-1} = \delta^{Y}_{k-1,k} = \delta^{Y}_{k,k-1}$, and so on. All these δ and θ quantities due to $X = 1$ and $Y = 1$ can be obtained from (39 a and b) and (40 a and b). Since the equations of continuity at the first step from either end of the beam contain only four unknowns instead of six, the possibility is presented of solving the system of such equations by the method of successive approximation. The same procedure can be followed when, instead of the cross section, there is a stepwise variation in the modulus of the foundation.

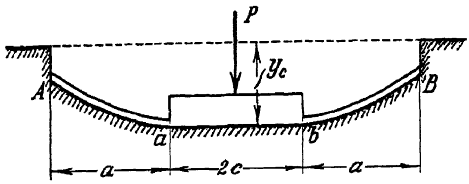

FIG. 87

As a special instance of the problem discussed above we can consider the situation shown in Figure 87. This figure represents column footings, where flexible flanges are attached on both sides of an infinitely rigid middle portion. Using formula (41a), we find, by considerations of statics, that the deflection of the middle portion will be

$$y_c = \frac{P\lambda}{k}\frac{1}{D}, \tag{75}$$

where

$$D = \frac{\text{Sinh}\, 2\lambda a + \sin 2\lambda a}{\text{Cosh}^2 \lambda a + \cos^2 \lambda a} + 2\lambda c.$$

The shearing force at points a and b can be obtained as $Q_a = -Q_b = \frac{1}{2}(P - 2cky_c)$. Having this value, we can compute the elastic line of the flanges as though we had a beam of length $2a$ loaded by a concentrated force $2Q_a$ at the middle.

29. *Continuous Variation*

Let us now consider the case in which the flexural rigidity of the beam EI_x and the modulus of the foundation k_x are both continuous differentiable functions of the variable x.*

* It will be assumed in the following discussion that only I varies as a function of x The procedure remains the same, however, if E is also considered to be a variable of x.

Substituting in the differential equation of bending

$$EI_x \frac{d^2y}{dx^2} = -M, \tag{a}$$

the relation between the bending moment M and the distributed reaction forces p of the foundation,

$$p = k_x y = \frac{d^2M}{dx^2}, \tag{b}$$

we obtain

$$\frac{d^2}{dx^2}\left(EI_x \frac{d^2y}{dx^2}\right) + k_x y = 0, \tag{c}$$

which, after the assigned differentiation has been carried out, takes the form:

$$\frac{d^4y}{dx^4} + \frac{2}{I_x}\frac{dI_x}{dx}\frac{d^3y}{dx^3} + \frac{1}{I_x}\frac{d^2I_x}{dx^2}\frac{d^2y}{dx^2} + \frac{k_x}{EI_x}y = 0. \tag{d}$$

If only the modulus of the foundation varies with x and I is constant, the equation above reduces to

$$\frac{d^4y}{dx^4} + \frac{k_x}{EI}y = 0. \tag{76'}$$

A differential equation of similar type can be established for variable I_x but constant k by differentiating (b) twice with respect to x and substituting the result in equation (a). Thus we get

$$\frac{d^4M}{dx^4} + \frac{k}{EI_x}M = 0. \tag{76''}$$

By comparing (76′) and (76″), it is seen that, whenever a solution is found for variable k and constant I, one can always find a corresponding case of variable I and constant k where the same type of solution will apply.

Rigorous solutions of differential equations of the types (76′) and (76″) can generally be derived in the form of power series, which are sometimes expressible in terms of Bessel functions. Approximate solutions for problems of this kind can be obtained most conveniently by either the Rayleigh–Ritz * analytical method or the Vianello–Stodola † graphical method.

The Rayleigh–Ritz method consists in assuming a set of functions each term of which satisfies the boundary conditions of the problem. The unknown

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* Lord Rayleigh, *Theory of Sound* (2d ed.; London, 1929), I, 111, 287; Walther Ritz, *Œuvres* (Paris, 1911), p. 265.

† Luigi Vianello, "Untersuchungen der Knickfestigkeit gerader Stabe," *Zeitschrift des Vereins deutscher Ingenieure*, 42, pt. 2 (1898), 1436; A. Stodola, *Steam and Gas Turbines* (New York, 1927), I, 449. See also A. Föppl, *Vorlesungen über technische Mechanik* (9th ed.; Leipzig, 1922), III, 264.

multipliers of the functions can be determined from the requirement that the total potential energy of the system have a minimum value in the equilibrium position. The procedure is the same as that applied in the case of constant I and k on page 75, where the elastic line was represented in the form of a trigonometric series.

It is seen from (76′) and (76″) that the problem of bending of a beam on an elastic foundation can be looked upon as that of finding the deflection curve which is proportional to the distribution of the loading of the beam, the proportionality factor being in general a function of x. The Vianello-Stodola method obtains this equilibrium position by successive approximations, assuming first the distribution of the loading and then determining graphically the deflection curve due to it. The deflection line leads to a corrected form of the corresponding loading, and the procedure, when repeated, will converge to the true solution, where the deflection curve, multiplied by the proportionality factor k_x in (b), will coincide with the loading distribution curve on the beam.

Both of these approximate methods can be used advantageously in practical applications. In the following sections, however, only those problems will be discussed in which a rigorous solution of the differential equation of the elastic line has been obtained, and approximate formulas will be mentioned only if they constitute special forms of a rigorous solution.

30. *Linearly Varying Moment of Inertia: the Circular Plate*

The most frequently occurring example of a beam with linearly varying moment of inertia is a wedge the width of which is a linear function of x (Fig. 88). Here, however, on account of the variable width of the beam, we shall have to deal also with a linearly varying value of k. Thus we shall have

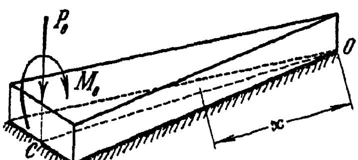

Fig. 88

$$I = I_0 x \qquad \text{and} \qquad k = k_0 b_0 x,$$

where b_0 and I_0 are the width and the moment of inertia of the beam at unit distance from the origin O, respectively, and where k_0 is the constant modulus of the foundation. A rigorous treatment of the bending of such a wedge-shaped beam will require the application of a system of polar coordinates. Thus the problem becomes analogous to that of a circular plate supported on an elastic foundation and subjected to a loading distributed symmetrically with respect to the center O. Under such conditions, the circular plate can be regarded as consisting of wedge-shaped elements, each of which is deformed in the same manner. The only difference between the beam and the plate problems is that in the latter the deformation of the cross section is restrained on account of the continuity of the material in the circumferential direction; thus opposing moments would act on the sides of each wedge element. From this point of view the beam problem can be considered as a limiting case of that of a bent circular plate; this will be the approach used in the following discussion.

The differential equation of bending of the plate can be obtained from the equilibrium conditions of an infinitesimal element of the plate (Fig. 89). Such an element will be acted upon, per unit length of its sides, by bending moments M_r and shearing forces Q_r in the radial and bending moments M_φ in the circumferential direction, while unit areas of its upper and lower surfaces will be subjected to a distributed loading q and a distributed reaction of the foundation p, respectively. Other force components vanish on account of the assumed symmetry of the loading.

A consideration of the equilibrium of the moments in the radial plane gives

$$\frac{dM_r}{dr} + \frac{M_r}{r} - \frac{M_\varphi}{r} - Q_r = 0, \tag{a}$$

while the equilibrium of the forces in the vertical direction requires that

$$\frac{dQ_r}{dr} + \frac{Q_r}{r} + q - p = 0. \tag{b}$$

Denoting by w the downward positive deflection, we can express the changes of curvature of the element due to bending in the radial and tangential directions respectively as

$$\kappa_r = \frac{d^2w}{dr^2} \quad \text{and} \quad \kappa = \frac{1}{r}\frac{dw}{dr}, \tag{c}$$

while the corresponding bending moments per unit length will be

$$M_r = -D(\kappa_r + \mu\kappa_\varphi) \quad \text{and} \quad M_\varphi = -D(\kappa_\varphi + \mu\kappa_r), \tag{d}$$

where $D = Eh^3/12(1 - \mu^2)$ is the flexural rigidity of the plate, h is its thickness and μ is Poisson's ratio for the plate material. By means of (c) the bending moments can be expressed as

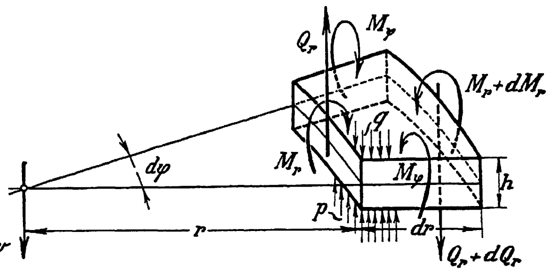

FIG. 89

$$\left.\begin{aligned} M_r &= -D\left(\frac{d^2w}{dr^2} + \frac{\mu}{r}\frac{dw}{dr}\right), \\ M_\varphi &= -D\left(\frac{1}{r}\frac{dw}{dr} + \mu\frac{d^2w}{dr^2}\right). \end{aligned}\right\} \tag{e}$$

Substituting these formulas in (a), we have for the shearing force

$$Q_r = -D\left(\frac{d^3w}{dr^3} + \frac{1}{r}\frac{d^2w}{dr^2} - \frac{1}{r^2}\frac{dw}{dr}\right). \tag{f}$$

The substitution of this expression in (b) gives

$$\frac{d^4w}{dr^4} + \frac{2}{r}\frac{d^3w}{dr^3} - \frac{1}{r^2}\frac{d^2w}{dr^2} + \frac{1}{r^3}\frac{dw}{dr} = \frac{q - p}{D},$$

where, according to our fundamental assumption, $p = k_0 w$. The distributed loading q can be eliminated from this equation by a change in the dependent variable, and thus the problem of the axially symmetrical bending of a circular plate on an elastic foundation reduces to the solution of the differential equation

$$\Delta_r^2 w + \nu^4 w = 0, \tag{77}$$

where Δ_r denotes the differential operator

$$\Delta_r = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} \quad \text{and} \quad \nu = \sqrt[4]{\frac{k_0}{D}}.$$

By putting $\nu r = \sqrt[4]{-1}\rho$, we can transform (77) into

$$\Delta_\rho^2 w - w = 0,$$

which in turn can be resolved into either of the two following forms:

$$\Delta_\rho(\Delta_\rho w + w) - (\Delta_\rho w + w) = 0$$

or

$$\Delta_\rho(\Delta_\rho w - w) + (\Delta_\rho w - w) = 0.$$

Hence it is seen that the solution of (77) can be obtained as the sum of the solutions for the two following differential equations:

$$\Delta_\rho w + w = 0 \qquad \text{and} \qquad \Delta_\rho w - w = 0.$$

The first of these equations is the same as the known Bessel equation with zero parameter value; the second can be transformed into the same form by changing the variable ρ to $i\psi$. The solution of the Bessel equation

$$\Delta_\rho w + w = \frac{d^2 w}{d\rho^2} + \frac{1}{\rho}\frac{dw}{d\rho} + w = 0$$

is known to be

$$w = A_1 J_0(\rho) + A_2 N_0(\rho),$$

where $J_0(\rho)$ and $N_0(\rho)$ are Bessel functions of the first kind and second kind (Neumann function) respectively, both of the zero order.

In a similar way we have the solution of

$$\Delta_\rho w - w = \frac{d^2 w}{d\rho^2} + \frac{1}{\rho}\frac{dw}{d\rho} - w = 0$$

in the form

$$w = A_3 J_0(i\rho) + A_4 N_0(i\rho).$$

Thus we obtain the complete solution of the original equation (77) as the sum of the two solutions above in terms of the original variable r:

$$w = A_1 J_0(\pm \nu r\sqrt{+i}) + A_2 N_0(\pm \nu r\sqrt{+i}) + A_3 J_0(\pm \nu r\sqrt{-i}) + A_4 N_0(\pm \nu r\sqrt{-i}).$$

For calculation it is more convenient if the solution is expressed in real functions of the argument νr, instead of in the form above. This can be achieved by introducing the following functions:*

$$\left.\begin{aligned} Z_1(\nu r) &= +\frac{1}{2}\,[J_0(\nu r\sqrt{+i}) + J_0(\nu r\sqrt{-i})],\\ Z_2(\nu r) &= -\frac{i}{2}\,[J_0(\nu r\sqrt{+i}) - J_0(\nu r\sqrt{-i})],\\ Z_3(\nu r) &= Z_1(\nu r) + \frac{i}{2}\,[N_0(\nu r\sqrt{+i}) - N_0(\nu r\sqrt{-i})],\\ Z_4(\nu r) &= Z_2(\nu r) + \frac{1}{2}\,[N_0(\nu r\sqrt{+i}) + N_0(\nu r\sqrt{-i})], \end{aligned}\right\} \tag{g}$$

when the solution of (77) will take the final form

$$w = C_1Z_1(\nu r) + C_2Z_2(\nu r) + C_3Z_3(\nu r) + C_4Z_4(\nu r). \tag{78}$$

The Z functions have the character of exponential waves; Z_1 and Z_2 increase rapidly with increasing argument, while Z_3 and Z_4 decrease as the argument increases. If we denote the argument by x, these functions can be written in the form of power series as follows:

$$\left.\begin{aligned} Z_1(x) &= 1 - \frac{\left(\frac{x}{2}\right)^4}{2!^2} + \frac{\left(\frac{x}{2}\right)^8}{4!^2} - \frac{\left(\frac{x}{2}\right)^{12}}{6!^2} + \cdots,\\ Z_2(x) &= -\frac{\left(\frac{x}{2}\right)^2}{1!^2} + \frac{\left(\frac{x}{2}\right)^6}{3!^2} - \frac{\left(\frac{x}{2}\right)^{10}}{5!^2} + \cdots,\\ Z_3(x) &= \frac{Z_1(x)}{2} - \frac{2}{\pi}\left[R_1 + \log_e \frac{\gamma x}{2}\cdot Z_2(x)\right],\\ Z_4(x) &= \frac{Z_2(x)}{2} + \frac{2}{\pi}\left[R_2 + \log_e \frac{\gamma x}{2}\cdot Z_1(x)\right], \end{aligned}\right\} \tag{79}$$

* These Z functions were first introduced by F. Schleicher and tabulated in his book, *Kreisplatten auf elastischer Unterlage* (Berlin, 1926). A table of these functions is also given at the end of the present work.

The Z functions can likewise be expressed as the real and imaginary parts of Bessel functions of the first kind, $J_0(x\sqrt{i})$, and third kind (Hankel function), $H_0^{(1)}(x\sqrt{i})$, in the following manner:

$$Z_1(x) = \operatorname{Re} J_0(x\sqrt{i}); \quad Z_3(x) = \operatorname{Re} H_0^{(1)}(x\sqrt{i});$$
$$Z_2(x) = \operatorname{Im} J_0(x\sqrt{i}); \quad Z_4(x) = \operatorname{Im} H_0^{(1)}(x\sqrt{i}).$$

The functions in these forms, together with their first derivatives are tabulated in E. Jahnke and F. Emde, *Tables of Functions* (3d ed.; Leipzig, 1938), pp. 246–257.

where

$$R_1 = \left(\frac{x}{2}\right)^2 - \frac{\varphi(3)}{3!^2}\left(\frac{x}{2}\right)^6 + \frac{\varphi(5)}{5!^2}\left(\frac{x}{2}\right)^{10} - \cdots,$$

$$R_2 = \frac{\varphi(2)}{2!^2}\left(\frac{x}{2}\right)^4 - \frac{\varphi(4)}{4!^2}\left(\frac{x}{2}\right)^8 + \frac{\varphi(6)}{6!^2}\left(\frac{x}{2}\right)^{12} - \cdots,$$

$$\varphi(n) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n},$$

and $\log_e \gamma = 0.577216$. Between the Z functions and their derivatives there exist the following relations:

$$\left.\begin{aligned} \frac{d^2Z_1(x)}{dx^2} &= Z_2(x) - \frac{1}{x}\frac{dZ_1(x)}{dx}, \\ \frac{d^2Z_2(x)}{dx^2} &= -Z_1(x) - \frac{1}{x}\frac{dZ_2(x)}{dx}, \\ \frac{d^2Z_3(x)}{dx^2} &= Z_4(x) - \frac{1}{x}\frac{dZ_3(x)}{dx}, \\ \frac{d^2Z_4(x)}{dx^2} &= -Z_3(x) - \frac{1}{x}\frac{dZ_4(x)}{dx}. \end{aligned}\right\} \tag{80}$$

In what follows we shall use, for derivatives of the $Z(x)$ functions with respect to the variable x, the simplified notations: $Z_1'(x) = dZ_1(x)/dx$, $Z_1''(x) = d^2Z_1(x)/dx^2$, and so on.

In limiting cases when $x \to 0$ or $x \to \infty$, the Z functions and their derivatives will approach the following asymptotic values:

	$\lim_{x\to 0} Z(x)$	$\lim_{x\to\infty} Z(x)$
$Z_1(x)$	$+1$	$+\alpha \cos \sigma$
$Z_2(x)$	$-\frac{x^2}{4}$	$-\alpha \sin \sigma$
$Z_3(x)$	$+\frac{1}{2}$	$+\beta \sin \tau$
$Z_4(x)$	$+\frac{2}{\pi}\log_e \frac{\gamma x}{2}$	$-\beta \cos \tau$

(81)

$$
\left.
\begin{array}{c|c|c}
 & \lim_{x\to 0} Z'(x) & \lim_{x\to\infty} Z'(x) \\
\hline
\dfrac{dZ_1(x)}{dx} & -\dfrac{x^3}{16} & +\dfrac{\alpha}{\sqrt{2}}(\cos\sigma - \sin\sigma) \\
\hline
\dfrac{dZ_2(x)}{dx} & -\dfrac{x}{2} & -\dfrac{\alpha}{\sqrt{2}}(\cos\sigma + \sin\sigma) \\
\hline
\dfrac{dZ_3(x)}{dx} & +\dfrac{x}{\pi}\log_e\dfrac{\gamma x}{2} & +\dfrac{\beta}{\sqrt{2}}(\cos\tau - \sin\tau) \\
\hline
\dfrac{dZ_4(x)}{dx} & +\dfrac{2}{\pi x} & +\dfrac{\beta}{\sqrt{2}}(\cos\tau + \sin\tau)
\end{array}
\right\} \qquad (81)
$$

where

$$
\alpha = \frac{1}{\sqrt{2\pi x}} e^{x/\sqrt{2}}, \qquad \beta = \sqrt{\frac{2}{\pi x}}\, e^{-x/\sqrt{2}},
$$

$$
\sigma = \frac{x}{\sqrt{2}} - \frac{\pi}{8}, \qquad \tau = \frac{x}{\sqrt{2}} + \frac{\pi}{8}.
$$

The Z functions have been tabulated up to the value 6 of the argument x. For $x > 6$ the asymptotic formulas for $x \to \infty$ can be used in the computation and will give an accuracy of five or more decimal places in the value of the functions.

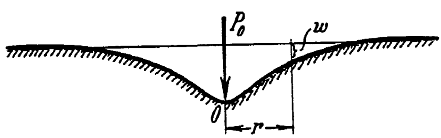

FIG. 90

This general solution for a circular plate on an elastic foundation will be applied now to a few practical problems. Let us first consider the case when a plate of infinite extension is subjected to a single concentrated force * P_0 acting at the origin O (Fig. 90). The four constants of integration in the general solution in (78) can be found from the boundary conditions of the problem. The consideration that at an infinite distance from the application of the load both w and dw/dr must vanish leads to $C_1 = 0$ and $C_2 = 0$. The condition that at the origin $dw/dr = 0$ can be satisfied only if $C_4 = 0$. The last of the constants C_3 can be obtained from the requirement that

$$
\lim_{r\to 0} (2r\pi Q_r) + P_0 = 0.
$$

* The solution for this case is due to H. Hertz, who was the first to analyze plates on elastic foundation in his paper "Über das Gleichgewicht schwimmender elastischer Platten," *Wiedemanns Annalen der Physik und Chemie*, v. 22 (1884), 449. Further references and discussion of a variety of problems connected with plates on elastic foundation can be found in S. Timoshenko's book, *Theory of Plates and Shells* (New York, 1940).

Substituting here from equation (f) (p. 101) the expression for Q_r and taking into account that $\lim_{\nu r \to 0} [\nu r Z_4'(\nu r)] = 2/\pi$, we have

$$C_3 = \frac{P_0}{4\nu^2 D},$$

and thus the elastic curve for an infinite plate under a single concentrated force P_0 can be expressed as

$$w = \frac{P_0}{4\nu^2 D} Z_3(\nu r), \tag{82}$$

where, as stated above, $\nu = \sqrt[4]{k_0/D}$ and $D = Eh^3/12(1 - \mu^2)$. The deflection under the load will be

$$w_0 = \frac{P_0}{8\nu^2 D}.$$

Substituting this solution for w in equations (e) and (f) (p. 101), we can calculate the values of M_r, M_φ and Q_r, and hence the stress distribution over the entire plate.

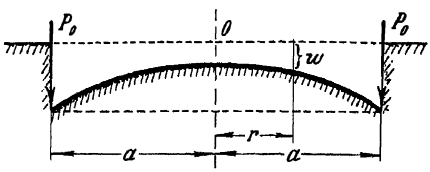

FIG. 91

Another problem of interest arises when a circular plate of radius a is subjected to uniformly distributed loading P_0 per unit run along the circumference (Fig. 91). Here the conditions that at $r = 0$, $dw/dr = 0$, and $Q_r = 0$ necessitate that $C_3 = C_4 = 0$ in the general solution stated in (78). Thus the equation of the deflection curve of the plate becomes

$$w = C_1 Z_1(\nu r) + C_2 Z_2(\nu r). \tag{83}$$

The two remaining integration constants can be determined from the conditions that at $r = a$, $M_r = 0$, and $Q_r = -P_0$ by using the corresponding expressions from equations (e) and (f) (p. 101). Thus we get

$$C_1 = -\frac{P_0 \nu}{k_0} \frac{Z_1(\nu a) + \frac{1-\mu}{\nu a} Z_2'(\nu a)}{Z_1(\nu a) Z_2'(\nu a) - Z_1'(\nu a) Z_2(\nu a) + \frac{1-\mu}{\nu a}[Z_1'^2(\nu a) + Z_2'^2(\nu a)]},$$

$$C_2 = -\frac{P_0 \nu}{k_0} \frac{Z_2(\nu a) + \frac{1-\mu}{\nu a} Z_1'(\nu a)}{Z_1(\nu a) Z_2'(\nu a) - Z_1'(\nu a) Z_2(\nu a) + \frac{1-\mu}{\nu a}[Z_1'^2(\nu a) + Z_2'^2(\nu a)]}.$$

If the same circular plate of radius a is loaded by uniformly distributed moments M_0 per unit length along the circumference (Fig. 92), since the conditions at $r = 0$ will be the same as in the previous case the expression for w will also take the same form:

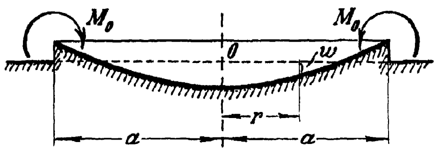

FIG. 92

$$w = D_1 Z_1(\nu r) + D_2 Z_2(\nu r), \tag{84}$$

but now the two integration constants will be determined so as to satisfy the requirements $M = M_0$ and $Q = 0$ for $r = a$. This will give, then,

$$D_1 = \frac{M_0 \nu^2}{k_0} \frac{Z_1'(\nu a)}{Z_1(\nu a)Z_2'(\nu a) - Z_1'(\nu a)Z_2(\nu a) + \frac{1-\mu}{\nu a}[Z_1'^2(\nu a) + Z_2'^2(\nu a)]},$$

$$D_2 = \frac{M_0 \nu^2}{k_0} \frac{Z_2'(\nu a)}{Z_1(\nu a)Z_2'(\nu a) - Z_1'(\nu a)Z_2(\nu a) + \frac{1-\mu}{\nu a}[Z_1'^2(\nu a) + Z_2'^2(\nu a)]}.$$

From the three loadings discussed above (Figs. 90–92) one can, by means of superposition, derive solutions for any plate of finite diameter loaded with a concentrated force at the middle and subjected to given boundary conditions along the circumference. The principle to be applied here is the same as that used in deriving finite beams from a beam of infinite length, and in the calculation the edge loadings P_0 and M_0 are to be regarded as end-conditioning forces.

After this treatment of the circular plate, let us now return to the discussion of the bending of the wedge-shaped beam shown in Figure 88. Putting $I_x = I_0 x$ and $k_x = k_0 b_0 x$ into the differential equation of flexure, equation (d) on page 99, gives

$$\frac{d^4 y}{dx^4} + \frac{2}{x}\frac{d^3 y}{dy^3} + \frac{b_0 k_0}{EI_0} y = 0.$$

The third derivative of y in this equation can be eliminated by the following change in the dependent variable:

$$z = y e^{\frac{1}{4}\int (2/x)dx} = y\sqrt{x},$$

which, if the ensuing lower derivatives of y are neglected, leads to the approximate differential equation of the deflection line below:

$$\frac{d^4 z}{dx^4} + \frac{b_0 k_0}{EI_0} z = 0,$$

the solution of which (expressed as the original variable y) may be written as

$$y = \frac{1}{\sqrt{x}} [e^{\lambda x}(C_1 \cos \lambda x + C_2 \sin \lambda x) + e^{-\lambda x}(C_3 \cos \lambda x + C_4 \sin \lambda x)], \qquad (85)$$

where

$$\lambda = \sqrt[4]{\frac{b_0 k_0}{4EI_0}}.$$

The same expression for the deflection line can be derived if in the general solution of the plate problem (see [78]) the asymptotic expressions of the Z functions for large values of the argument are substituted from (81). This proves that the approximate solution above will be very accurate for large ($\lambda x > 6$) values; incidentally, such cases are most frequent among the applications of this theory to foundation plates and similar structures.

In calculating circular foundation plates for tanks or containers the displacements along the edge of the plate are of primary importance. If we take $b_0 = 1$ and $I_0 = h^3/12(1 - \mu^2)$, the asymptotic formulas furnish the following simple expressions for these so-called *influence factors* due to uniformly distributed edge loadings P_0 and M_0 per unit length of the circumference of the plate of radius a:

Deflection of the edge due to P_0 lbs./in.:

$$w_{P_0} = \frac{P_0 \lambda}{k_0} \frac{2\lambda a - 1}{\lambda a - 1};$$

Rotation of the edge due to P_0 lbs./in.:

$$\theta_{P_0} = -\frac{2P_0 \lambda^2}{k_0} \frac{\lambda a}{\lambda a - 1};$$

Deflection of the edge due to M_0 in. lbs./in.:

$$w_{M_0} = -\frac{2M_0 \lambda^2}{k_0} \frac{\lambda a}{\lambda a - 1};$$

Rotation of the edge due to M_0 in. lbs./in.:

$$\theta_{M_0} = \frac{4M_0 \lambda^3}{k_0} \frac{\lambda a}{\lambda a - 1}.$$

31. *Linearly Varying Modulus of Foundation*

Consider a beam of constant cross section supported on an elastic foundation, the modulus of which varies according to a linear law $k_x = k_A - cx$, where k_A lbs./in.^{2} and c lbs./in.^{3} are constants and x denotes the distance along the beam from the fixed point A.

Using the expression above for k_x in the differential equation of bending, we have

$$\frac{EI}{k_A}\frac{d^4y}{dx^4} + \frac{k_A - cx}{k_A}\,y = 0. \tag{a}$$

By introducing a new variable,

$$\xi = \frac{k_A - cx}{k_A}, \tag{b}$$

we have

$$\frac{d^4y}{dx^4} = \frac{d^4y}{d\xi^4}\left(\frac{d\xi}{dx}\right)^4 = \frac{d^4y}{d\xi^4}\left(\frac{c}{k_A}\right)^4,$$

and thus (a) takes the form

$$\frac{d^4y}{d\xi^4} + \alpha\xi y = 0, \tag{86}$$

where

$$\alpha = \frac{k_A^5}{c^4 EI}.$$

In seeking the solution of this differential equation let us first substitute in it $y = \xi^m$. This leads to the expression

$$m(m - 1)(m - 2)(m - 3)\xi^{m-4} + \alpha\xi^{m+1} = 0,$$

from which we can conclude that there will be four series of the type

$$y = a_0\xi^m + a_1\xi^{m+5} + \cdots + a_k\xi^{m+5k} + \cdots, \tag{c}$$

in which $m = 0, 1, 2$, and 3, respectively, that will satisfy (86); consequently, the general solution of that differential equation will consist of the sum of these four series, each one multiplied by a different integration constant.

By putting the general form of these series, equation (c), into (86) we get the relation between any two consecutive coefficients a_k and a_{k+1} as

$$a_{k+1} = -\frac{\alpha}{(m + 5k + 5)(m + 5k + 4)(m + 5k + 3)(m + 5k + 2)}\,a_k.$$

By means of this recursion formula the four series y_1, y_2, y_3, and y_4 which comprise the general solution

$$y = C_1y_1 + C_2y_2 + C_3y_3 + C_4y_4 \tag{87a}$$

can be written as

$$y_1 = 1 - \frac{\alpha}{5!}\xi^5 + \frac{6\alpha^2}{10!}\xi^{10} - \frac{6\cdot 11\cdot\alpha^3}{15!}\xi^{15} + \frac{6\cdot 11\cdot 16\cdot\alpha^4}{20!}\xi^{20} - \cdots,$$

$$y_2 = \xi - \frac{2\alpha}{6!}\xi^6 + \frac{2\cdot 7\cdot\alpha^2}{11!}\xi^{11} - \frac{2\cdot 7\cdot 12\cdot\alpha^3}{16!}\xi^{16} + \frac{2\cdot 7\cdot 12\cdot 17\cdot\alpha^4}{21!}\xi^{21} - \cdots,$$

$$y_3 = \frac{\xi^2}{2!} - \frac{3\alpha}{7!}\xi^7 + \frac{3\cdot 8\cdot\alpha^2}{12!}\xi^{12} - \frac{3\cdot 8\cdot 13\cdot\alpha^3}{17!}\xi^{17} + \frac{3\cdot 8\cdot 13\cdot 18\cdot\alpha^4}{22!}\xi^{22} - \cdots,$$

$$y_4 = \frac{\xi^3}{3!} - \frac{4\alpha}{8!}\xi^8 + \frac{4\cdot 9\cdot\alpha^2}{13!}\xi^{13} - \frac{4\cdot 9\cdot 14\cdot\alpha^4}{18!}\xi^{18} + \frac{4\cdot 9\cdot 14\cdot 19\cdot\alpha^4}{23!}\xi^{23} - \cdots.$$

From the expression above for the deflection line one can obtain by successive differentiation the angular deflection θ, bending moment M, and shearing force Q at any point of the beam as follows:

$$\left.\begin{aligned}\theta &= \frac{dy}{dx} = \frac{dy}{d\xi}\frac{d\xi}{dx} = -\frac{c}{k_A}\frac{dy}{d\xi},\\ M &= -EI\frac{d^2y}{dx^2} = -EI\frac{d^2y}{d\xi^2}\left(\frac{d\xi}{dx}\right)^2 = -EI\left(\frac{c}{k_A}\right)^2\frac{d^2y}{d\xi^2},\\ Q &= -EI\frac{d^3y}{dy^3} = -EI\frac{d^3y}{d\xi^3}\left(\frac{d\xi}{dx}\right)^3 = EI\left(\frac{c}{k_A}\right)^3\frac{d^3y}{d\xi^3}.\end{aligned}\right\} \quad (87\text{ b–d})$$

Hence it is seen that the integration constants in the general solution represent the end conditions y_0, θ_0, M_0, and Q_0 of the beam at the point where $\xi = 0$, that is,

$$\left.\begin{aligned}[y]_{\xi=0} &= C_1 = y_0,\\ \left[\frac{dy}{d\xi}\right]_{\xi=0} &= C_2 = -\frac{k_A}{c}\theta_0,\\ \left[\frac{d^2y}{d\xi^2}\right]_{\xi=0} &= C_3 = -\left(\frac{k_A}{c}\right)^2\frac{M_0}{EI},\\ \left[\frac{d^3y}{d\xi^3}\right]_{\xi=0} &= C_4 = \left(\frac{k_A}{c}\right)^3\frac{Q_0}{EI}.\end{aligned}\right\} \quad (88\text{ a–d})$$

As a numerical example of the solution developed above let us consider the problem shown in Figure 93, where the constants of the beam are assumed to be: $I = 144$ in.^{4}; $E = 2.5 \times 10^6$ lbs./in.^{2}; $k_A = 700$ lbs./in.^{2}; $c = 5$ lbs./in.^{3}; $l = 120$ in.; $\alpha = k_A^5/c^4EI = 746.98$. The four integration constants will be defined by the end conditions

$$[M]_{x=0} = 0, \qquad [M]_{x=l} = 0,$$

$$[Q]_{x=0} = -P, \qquad [Q]_{x=l} = P.$$

These end conditions can be written by means of (88 a–d) as follows:

$$-45.88\,C_1 - 41.52\,C_2 - 14.08\,C_3 - 2.66\,C_4 = 0,$$

$$-124.49\,C_1 - 8.89\,C_2 + 342.62\,C_3 + 48.98\,C_4 = 0,$$

$$+234.02\,C_1 - 66.18\,C_2 - 57.95\,C_3 - 18.60\,C_4 = -\frac{P}{EI}\left(\frac{k_A}{c}\right)^3,$$

$$-7.623\,C_1 - 0.723\,C_2 - 0.039\,C_3 + 0.998\,C_4 = +\frac{P}{EI}\left(\frac{k_A}{c}\right)^3.$$

The solution of these four simultaneous equations for the C's gives

$$C_1 = +0.0345 \frac{P}{EI}\left(\frac{k_A}{c}\right)^3,$$

$$C_2 = -0.0609 \frac{P}{EI}\left(\frac{k_A}{c}\right)^3,$$

$$C_3 = -0.1619 \frac{P}{EI}\left(\frac{k_A}{c}\right)^3,$$

$$C_4 = +1.2113 \frac{P}{EI}\left(\frac{k_A}{c}\right)^3.$$

Hence for the deflection line of the beam we have the expression

$$y = \frac{P}{EI}\left(\frac{k_A}{c}\right)^3 (0.0345 - 0.0609\,\xi - 0.0810\,\xi^2 + 0.2019\,\xi^3 - 0.2148\,\xi^5 + 0.1264\,\xi^6 + 0.0720\,\xi^7 - 0.0898\xi^8 + 0.0319\,\xi^{10} - 0.0119\,\xi^{11} - 0.0045\,\xi^{12} + 0.0039\,\xi^{13} - 0.0007\,\xi^{15} - 0.002\,\xi^{16} + \cdots).$$

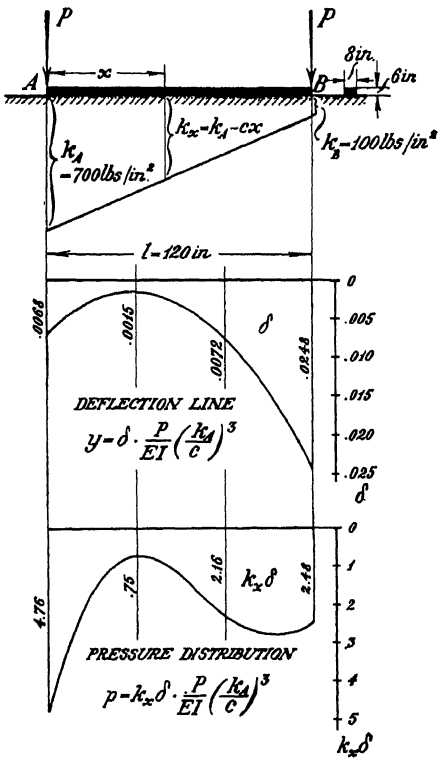

Fig. 93

The deflection line calculated by this formula is shown in Figure 93, together with the corresponding pressure distribution in the foundation, $p = k_x y$.

It has been shown in § 29, by way of comparing (76′) and (76″), that whenever a general solution of a problem involving constant I and variable k_x is known, it will also give the solution of a corresponding case of constant k and variable I_x, which I_x, however, will be inversely proportional to the K_x of the first case. Hence we know that the solution discussed above for constant I and linearly varying k_x can be used where k is constant and $I_x = (1/x)I_0$ varies inversely (hyperbolically) with x.

The differential equation for the bending moment M will now have the same form, according to (76″), as had the differential equation of the deflection line in the previous problem; it will be

$$\frac{d^4M}{dx^4} + \frac{k}{EI_0}\,xM = 0, \qquad (d)$$

and its solution can be written readily in the form found for y in (87a).

In foundation works beams are frequently haunched at points of concentrated loadings, in which event the distribution of the flexural rigidity along the beam can be closely approximated by the assumption of a hyperbolic variation

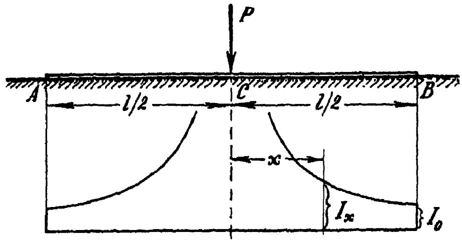

FIG. 94

in I. In considering such a finite beam with a concentrated force at the middle (Fig. 94) the calculation will be made more convenient if we introduce a new variable,

$$\xi = \frac{2x}{l},$$

taking at the same time $I_x = (1/\xi)I_0$, where I_0 denotes the moment of inertia of the beam at the place $\xi = 1$, that is, at the ends. In terms of this new variable, equation (d) will become

$$\frac{d^4M}{d\xi^4} + \alpha\xi M = 0. \tag{89}$$

where

$$\alpha = \frac{kl^4}{16EI_0}.$$

The solution of this differential equation can be written at once, using the same scheme as in (87a). Expressions for Q, y, and θ can be obtained by successive differentiation of the general solution for M. The series will rapidly converge in every case, since the new variable is always less than, or equal to, unity.

32. *Beam of Linearly Varying Depth*

If the depth of the beam is a linear function of x, its flexural rigidity will vary proportionally to the third power of x, $I_x = I_0x^3$, and thus the differential equation of the deflection line, according to epuation (d) on page 99, will be, if the modulus of foundation is constant,

$$x^3\frac{d^4y}{dx^4} + 6x^2\frac{d^3y}{dx^3} + 6x\frac{d^2y}{dx^2} + \frac{k}{EI_0}y = 0. \tag{90}$$

A substitution of $y = x^m$ in this equation gives

$$m^2(m^2 - 1)x^{m-1} + \frac{k}{EI_0}x^m = 0,$$

which suggests that the differential equation can be satisfied with power series of the form

$$y = a_0x^m + a_1x^{m+1} + \cdots + a_kx^{m+k} + \cdots, \tag{a}$$

where m is any of the four roots of the equation $m^2(m^2 - 1) = 0$.

The discussion below applies not to the general solution, but to the solution of the particular problem illustrated in Figure 95, where the beam can be regarded as an idealized form of a column footing. The major problem in such a structure is to find its effective length, that is, the limiting case when at the ends of the beam the deflection and, consequently, the pressure in the foundation vanish. This end condition of $y = 0$ at $x = 0$, in addition to the $[M]_{x=0}$ and $[Q]_{x=0}$ requirements of a free end, is satisfied only by that series, equation (a), of the general solution which belongs to the root $m = +1$.

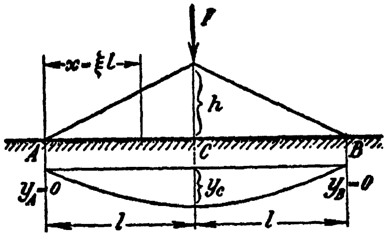

FIG. 95

Hence the solution of the problem, after a convenient change in the independent variable $x = \xi l$, will be given by the series

$$y = a_0\xi + a_1\xi^2 + a_2\xi^3 + \cdots + a_k\xi^{k+1} + \cdots,$$

which, when substituted in the differential equation leads to the recursion formula

$$a_{k+1} = -\frac{\alpha}{(k+1)(k+2)^2(k+3)}\,a_k\,, \tag{b}$$

where

$$\alpha = \frac{kl}{EI_0}.$$

By means of this formula the solution can be written as

$$y = a_0\left(\xi - \frac{\alpha}{(1\cdot 2^2\cdot 3)}\xi^2 + \frac{\alpha^2}{(1\cdot 2^2\cdot 3)(2\cdot 3^2\cdot 4)}\xi^3 - \frac{\alpha^3}{(1\cdot 2^2\cdot 3)(2\cdot 3^2\cdot 4)(3\cdot 4^2\cdot 5)}\xi^4 + \cdots\right). \tag{c}$$

This solution will have to satisfy, by a proper choice of α and a_0, two more end conditions involved in the problem, namely, the conditions that at the center of the beam the angular deflection $\theta = 0$ and the shearing force $Q = -P/2$.

The condition that at $\xi = 1$, $\theta = \frac{dy}{dx} = \frac{dy}{ld\xi} = 0$ requires that

$$0 = 1 - \frac{\alpha}{6} + \frac{\alpha^2}{12\cdot 24} - \frac{\alpha^3}{12\cdot 72\cdot 60} + \cdots,$$

which gives the value of $\alpha = kl/EI_0 = 7$ or $I_0 = kl/7E$. Since, according to our assumption, $I_0 = bh^3/12l^3$, where h is the height of the beam at the center (Fig. 95) and b is its constant width, the effective length of the beam will be reached only if the height h is chosen in such a manner as to satisfy the following relation: *

$$h = l\sqrt[3]{\frac{12kl}{7Eb}}. \tag{d}$$

* This condition was established by Adolf Francke, "Einiges über Fundamente," *Schweizerische Bauzeitung*, 35 (1900), 145.

Putting $\alpha = 7$ into (c), we can determine the last constant a_0 from the condition

$$[Q]_{\xi=1} = EI_0\left[\frac{d^3y}{d\xi^3} + 3\frac{d^2y}{d\xi^2}\right]_{\xi=1} = -\frac{P}{2},$$

which gives

$$a_0 = \frac{1.6\,P}{kl}. \tag{e}$$

By substituting these a_0 and α values in (c) we have the final form of the deflection line

$$y = \frac{1.6\,P}{kl}\left(\xi - \frac{7}{12}\xi^2 + \frac{7^2}{12\cdot 72}\xi^3 - \frac{7^3}{12\cdot 72\cdot 240}\xi^4 + \frac{7^4}{12\cdot 72\cdot 240\cdot 600}\xi^5 - \cdots\right), \tag{90}$$

in which case the maximum pressure at the center C will be

$$p_{\max} = ky_c = \frac{0.75\,P}{l}. \tag{f}$$

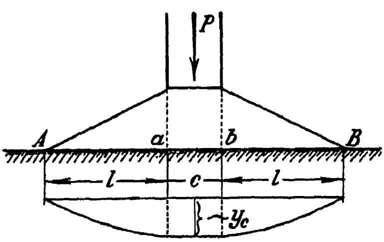

FIG. 96

These formulas can be applied also when the central part of the beam is rigid and not deformable (Fig. 96). Denoting by Q_l that part of the P loading which is transmitted in the form of a shearing force to one of the side portions Aa or bB, we can write from (f) that

$$p_{\max} = \frac{0.75\cdot 2Q_l}{l}.$$

Since equilibrium requires that $2Q_l + p_{\max}\cdot c = P$, we have

$$2Q_l = \frac{P}{1 + 0.75\,\dfrac{c}{l}}.$$

Knowing Q_l, we can calculate the deflection line by using the solution obtained above in (90).

33. *Cylindrical Tank with Linearly Varying Wall Thickness*

Let us assume a cylindrical container with linearly increasing wall thickness toward the base and subjected to an internal hydraulic pressure:* Choosing

* This problem was first investigated by H. Reissner, "Über die Spannungsverteilung in zylindrischen Behälterwänden," *Beton und Eisen*, 7 (1908), 150.

See also W. Flügge, *Statik und Dynamik der Schalen* (Berlin, 1934), and S. Timoshenko, *op. cit.* (see p. 105).

the origin of the x coordinates as shown in Figure 97, we can express the linear law of variation of the wall thickness as $t = \alpha x$, α being a numerical factor. Accordingly, the effective moment of inertia of a longitudinal element of unit width will be $I = \alpha^3 x^3/[12(1 - \mu^2)]$. The modulus of foundation for such a longitudinal element is, from (25), $k = E\alpha x/R^2$, and the hydraulic loading per unit length of this element is $q = -\gamma(x - x_0)$, where γ is the specific weight of the liquid in the container.

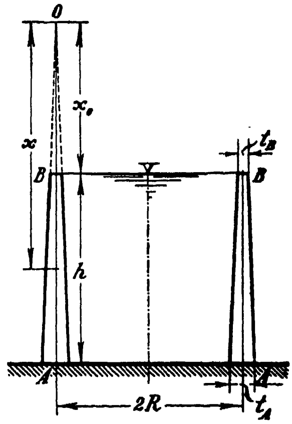

FIG. 97

Putting these expressions into the differential equation for bending,

$$\frac{d^2}{dx^2}\left(EI\frac{d^2y}{dx^2}\right) + ky = q,$$

we have, by suitable arrangements of the terms,

$$\frac{d^2}{dx^2}\left(x^3\frac{d^2y}{dx^2}\right) + \frac{12(1 - \mu^2)}{\alpha^2R^2}xy = -\frac{12(1 - \mu^2)}{E\alpha^3}\gamma(x - x_0). \tag{a}$$

It is seen that a particular integral of this differential equation will be

$$y_1 = -\frac{R^2\gamma}{E\alpha}\frac{x - x_0}{x}. \tag{b}$$

This particular integral represents the deformation which takes place when the cylindrical wall is not restrained at the base and is free to expand because of the action of the internal hydraulic loading. The deformation will result at the end A, where $x = x_0 + h$, in a radial displacement

$$y_A = -\frac{R^2\gamma}{E\alpha}\frac{h}{(x_0 + h)} \tag{c}$$

and in an angular deflection

$$\theta_A = -\frac{R^2\gamma}{E\alpha}\frac{x_0}{(x_0 + h)^2}. \tag{d}$$

As a result of this deformation the end conditions at the base ($y_A = 0$ and $\theta_A = 0$, for instance, if the base is rigidly fixed) are not fulfilled. In order to satisfy the end conditions we must apply around the base circle distributed forces and moments. This part of the solution, implying bending deformations, can be obtained from the homogeneous form of (a), that is, from

$$\frac{1}{x}\frac{d^2}{dx^2}\left(x^3\frac{d^2y}{dx^2}\right) + \frac{12(1 - \mu^2)}{\alpha^2R^2}y = 0. \tag{e}$$

It can be shown by differentiation that

$$\frac{1}{x}\frac{d^2}{dx^2}\left(x^3\frac{d^2y}{dx^2}\right) = \frac{1}{x}\frac{d}{dx}\left\{x^2\frac{d}{dx}\left[\frac{1}{x}\frac{d}{dx}\left(x^2\frac{dy}{dx}\right)\right]\right\}$$

which makes it possible to resolve (e) into two conjugate equations of the second order:

$$\frac{1}{x}\frac{d}{dx}\left(x^2\frac{dy}{dx}\right) \pm i\lambda^2 y = 0, \tag{91}$$

where

$$\lambda = \sqrt[4]{\frac{12(1-\mu^2)}{\alpha^2 R^2}}.$$

Taking the first equation in (91), the one with the plus sign, and carrying out the assigned differentiation, we have

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + i\lambda^2 y = 0.$$

Now putting $y = \eta/\sqrt{x}$ and $\xi = 2\lambda\sqrt{x}$ and using the notation $\xi\sqrt{i} = \omega$, we find that the equation above becomes

$$\omega^2\frac{d^2\eta}{d\omega^2} + \omega\frac{d\eta}{d\omega} + (\omega^2 - 1)\,\eta = 0,$$

which is a Bessel equation with a parameter value of unity; its solution is known to be

$$\eta = AJ_1(\omega) + BN_1(\omega),$$

where J_1 and N_1 are Bessel functions of the first and second kind respectively, both of the first order.

The second equation in (91), the one with the minus sign, can be transformed in the same manner, yielding the solution

$$\eta = CJ_1(i\omega) + DN_1(i\omega).$$

Consequently, the general solution of (91), as the sum of these two solutions' can be written as

$$y = \frac{1}{\sqrt{x}}\{AJ_1(\xi\sqrt{+i}) + BN_1(\xi\sqrt{+i}) + CJ_1(\xi\sqrt{-i}) + DN_1(\xi\sqrt{-i})\}.$$

By the use of the known relations

$$J_1(\omega) = -\frac{dJ_0(\omega)}{d\omega} \quad \text{and} \quad N_1(\omega) = -\frac{dN_0(\omega)}{d\omega}$$

this solution can be expressed by means of Bessel functions of the zero order, reducing thus to a form similar to that encountered in the circular plate problem in § 30. By the same method employed there, that is, by introducing Schleicher's Z functions (equation [g], p. 103) of the real argument $\xi = 2\lambda\sqrt{x}$, the general solution of (91) can be written in the final form

$$y_x = \frac{1}{\sqrt{x}}\{C_1Z_1'(\xi) + C_2Z_2'(\xi) + C_3Z_3'(\xi) + C_4Z_4'(\xi)\}, \tag{92a}$$

where the prime denotes differentiation with respect to the argument ξ.

By successive differentiations one can derive from the equation above the expressions for the angular deflection θ_x, bending moment M_x, and shearing force Q_x:

$$\theta_x = \frac{dy}{dx} = \frac{1}{2x\sqrt{x}}\{C_1[\xi Z_2(\xi) - 2Z_1'(\xi)] - C_2[\xi Z_1(\xi) + 2Z_2'(\xi)] + C_3[\xi Z_4(\xi) - 2Z_3'(\xi)] - C_4[\xi Z_3(\xi) + 2Z_4'(\xi)]\}, \tag{92b}$$

$$M_x = EI\frac{d^2y}{dx^2} = \frac{E\alpha^3}{48(1-\mu^2)}\sqrt{x}\,\{C_1[\xi^2Z_2'(\xi) - 4\xi Z_2(\xi) + 8Z_1'(\xi)] - C_2[\xi^2Z_1'(\xi) - 4\xi Z_1(\xi) - 8Z_2'(\xi)], + C_3[\xi^2Z_4'(\xi) - 4\xi Z_4(\xi) + 8Z_3'(\xi)], - C_4[\xi^2Z_3'(\xi) - 4\xi Z_3(\xi) - 8Z_4'(\xi)]\}, \tag{92c}$$

$$Q_x = EI\frac{d^3y}{dx^3} = -\frac{E\alpha^2\lambda^2}{24(1-\mu^2)}\sqrt{x}\,\{C_1[\xi Z_1(\xi) + 2Z_2'(\xi)] + C_2[\xi Z_2(\xi) - 2Z_1'(\xi)] + C_3[\xi Z_3(\xi) + 2Z_4'(\xi)] + C_4[\xi Z_4(\xi) - 2Z_3'(\xi)]\}. \tag{92d}$$

The hoop forces N are also known from the radial deflection y, since, according to equation (a) on page 31,

$$N = \frac{E\alpha x}{R}y, \tag{92e}$$

while the bending moment in the circumferential direction is obtained from equation (c) on page 31 as

$$M_c = \mu M_x. \tag{92f}$$

Equations (92 a–f) represent the complete solution for the problem of the axially symmetrical bending of a cylindrical tube with linearly varying wall thickness. The four integration constants occurring in these equations can be determined from the four end conditions, which, two at each end, are known in advance in every problem.

Applying (92 a–f) to the problem in Figure 97 we find that here the conditions for a fixed base at A and a free end at B require that, for

$$B \text{ at } x = x_0, \qquad M_x = 0 \quad \text{and} \quad Q_x = 0.$$

and for

$$A \text{ at } x = x_0 + h, \quad y_x = -y_A \quad \text{and} \quad \theta_x = -\theta_A,$$

where y_A and θ_A denote the expressions given by equations (c) and (d) on page 115.

Since the functions Z_1, Z_2, and their derivatives increase with increasing values of the argument, while Z_3, Z_4, and their derivatives decrease at the same time, the constants C_1 and C_2 will be determined mainly by the conditions existing at the lower end of the cylinder, while C_3 and C_4 will be determined largely by those at the upper end. The tables at the end of the present work give the values of the Z functions and their first derivatives up to the value of the argument $\xi = 6$. If $\xi > 6$, then the asymptotic formulas of the Z functions (see pp. 104–105) can be used in the calculation.

Consider, for example, a cylindrical tank with these dimensions: $R = 360$ in.; $h = 312$ in.; $t_A = 14$ in. (wall thickness at the base); $t_B = 3\frac{1}{2}$ in. (wall thickness at the top). The material is reinforced concrete $E = 4.25 \times 10^6$ lbs./in.^{2}, $\mu = 0.25$, and the tank is filled with water, specific gravity $\gamma = 0.0361$ lbs./in.^{3}

It is seen that the dimensions of the tank are the same as those in the example on page 89, except that, whereas there the wall thickness was constant, $t = 14$ in., here the wall tapers off from that value at the base to $3\frac{1}{2}$ in. at the upper end.

The dimensions assumed above give

$$x_0 = \frac{ht_B}{t_A - t_B} = 104 \text{ in.}, \quad \alpha = \frac{t_B}{x_0} = 0.0336,$$

and

$$\lambda = \sqrt[4]{\frac{12(1 - \mu^2)}{\alpha^2 R^2}} = 0.526.$$

Since even for the upper end $\xi_0 = 2\lambda\sqrt{x_0} = 10.72 > 6$, it is permissible to take $Z_3 = Z_4 = 0$ and use for the rest of the calculation the asymptotic expressions for Z_1 and Z_2. Thus the conditions to be satisfied at the base of the tank can be written as

$$\frac{1}{\sqrt{x_0 + h}}\left\{\frac{e^{\xi/\sqrt{2}}}{\sqrt{2\pi\xi}}\left[C_1 \cos\left(\frac{\xi}{\sqrt{2}} + \frac{\pi}{8}\right) - C_2 \sin\left(\frac{\xi}{\sqrt{2}} + \frac{\pi}{8}\right)\right]\right\}_{\xi=2\lambda\sqrt{x_0+h}} = \frac{R^2\gamma}{E\alpha}\frac{h}{x_0 + h},$$

$$\frac{1}{2\sqrt{(x_0 + h)^3}}\left\{\frac{e^{\xi/\sqrt{2}}}{\sqrt{2\pi\xi}}\left[C_1\left(-\xi \sin\left(\frac{\xi}{\sqrt{2}} - \frac{\pi}{8}\right) - 2\cos\left(\frac{\xi}{\sqrt{2}} + \frac{\pi}{8}\right)\right)\right.\right.$$
$$\left.\left. - C_2\left(\xi \cos\left(\frac{\xi}{\sqrt{2}} - \frac{\pi}{8}\right) - 2\sin\left(\frac{\xi}{\sqrt{2}} + \frac{\pi}{8}\right)\right)\right]\right\}_{\xi=2\lambda\sqrt{x_0+h}} = \frac{R^2\gamma}{E\alpha}\frac{x_0}{(x_0 + h)^2}.$$

Solving these equations for C_1 and C_2, we have

$$C_1 = -0.862C \quad \text{and} \quad C_2 = -0.958C,$$

where

$$C = \frac{R^2\gamma}{E\alpha}\,\frac{h}{\sqrt{x_0+h}}\,[e^{-\xi/\sqrt{2}}\;\sqrt{2\pi\xi}]_{\xi=2\lambda\sqrt{x_0+h}}.$$

Substituting these values of the integration constants in (92 c–d) and putting numerical values in for the dimensions, we get the bending moment M_A and the shearing force Q_A acting per unit length of the circumference at the base of the tank:

$$M_A = 13{,}860 \text{ in. lbs./in.} \qquad \text{and} \qquad Q_A = 527 \text{ lbs./in.}$$

If we compare these results with those obtained for the same container but with constant $t = 14$ in. wall thickness ($M_A = 14{,}000$ in. lbs./in. and $Q_A = 564$ lbs./in., p. 90), it is seen that the tapering of the wall had no appreciable effect on the magnitude of the maximum stresses around the base of the tank; thus the use of this structural feature proves to be truly economical.

34. *Conical Shell*

Problems involving axially symmetrical deformation of conical shells can also be reduced, in a way analogous to that used for cylindrical shells, to our fundamental problem of bending of a beam on an elastic foundation. The wall of the conical shell will be regarded as consisting of a large number of longitudinal tapered beams supported on transverse elastic rings whose diameter increases in proportion to the distance from the apex of the cone. This picture, though representing closely the actual conditions in the shell, omits one of the stress components, namely, that due to the circumferential bending of the hoop rings. On account of the lateral contraction of the longitudinal elements, bending moments will be set up in the transverse rings, and such moments, because of the tapering, will have components in the principal plane of the longitudinal beams. These components, however, which are taken into account in the rigorous theory for thin shells,* have significance only in extremely flat shells, which could be well calculated as circular plates. With such exceptions the rest of the shells can be conveniently analyzed by the present approximate method. The degree of accuracy and the range of applicability of this method will be discussed later on in this section.

Let us consider a conical shell with constant wall thickness δ small in proportion to the other dimensions of the shell (Fig. 98). The modulus of the foundation furnished by the hoop rings, per unit length of circumference of the rings, is, according to (25), $k_0 = Et/R^2$, where $R = x \sin\alpha$ and where $t = \delta/\cos\alpha$ is the thickness of the rings in the direction normal to the axis of the cone. Hence

* The most comprehensive study of the exact solution for conical shells has been published by F. Dubois, *Über die Festigkeit der Kegelschale* (Dissertation, Zürich, 1917). See also Flügge, *op. cit.* (see p. 114), and Timoshenko, *op. cit.* (see p. 105).

the modulus per unit length of the longitudinal beams will be $k = bk_0$, where $b = b_0 x$ is the width of the beams increasing linearly with the distance from the apex. Thus we have

$$k = b_0 x k_0 = \frac{b_0 E\delta}{x \sin^2 \alpha \cos \alpha}.$$

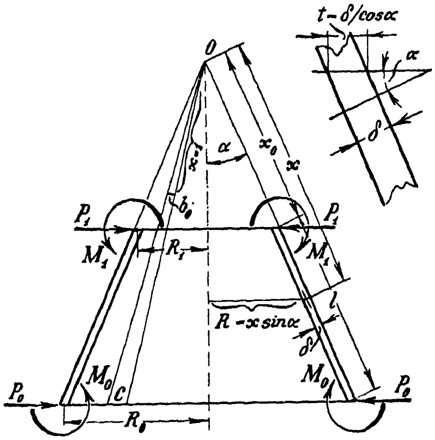

FIG. 98

At the same time, the flexural rigidity of one beam will be

$$EI = \frac{b_0 x E\delta^3}{12 \cos^3 \alpha},$$

the effect of Poisson's ratio of the material being omitted on account of the nature of the approximation mentioned above.

Putting these values of k and EI into the differential equation of bending

$$\frac{d^2}{dx^2}\left(EI \frac{d^2 y}{dx^2}\right) + ky = 0,$$

we can write the result in the form

$$\frac{x}{\cos \alpha} \frac{\delta^2}{12} \sin^2 \alpha \frac{d^2}{dx^2}\left(\frac{x}{\cos \alpha} \frac{d^2 y}{dx^2}\right) = -y. \tag{a}$$

Putting

$$\frac{x}{\cos \alpha} \frac{d^2 y}{dx^2} = F, \tag{b}$$

we have

$$\frac{x}{\cos \alpha} \rho \frac{d^2 F}{dx^2} = -y,$$

where

$$\rho = \frac{\delta^2}{12} \sin^2 \alpha.$$

Assuming now that $F = \omega y$, we find that these equations become

$$\frac{x}{\cos \alpha} \frac{d^2 y}{dx^2} = \omega y$$

and

$$\frac{x}{\cos \alpha} \rho\omega \frac{d^2 y}{dx^2} = -y.$$

giving

$$\rho\omega^2 y = -y \qquad \text{or} \qquad \omega = \pm\frac{i}{\sqrt{\rho}}.$$

Hence we have $F = \pm(i/\sqrt{\rho})y$, which, when substituted in (b) yields

$$x\frac{d^2 y}{dx^2} \pm i\lambda^2 y = 0, \tag{c}$$

where

$$\lambda = \sqrt[4]{\frac{12}{\delta^2 \tan^2 \alpha}}.$$

Thus, as a result of the transformations above, the original differential equation of the fourth order (a) is resolved into two simultaneous conjugate equations of the second order. By putting $\zeta = \xi\sqrt{i}$, where $\xi = 2\lambda\sqrt{x}$, as suggested by E. Meissner,* we can again change these resulting equations to a form

$$\frac{d^2 y}{d\zeta^2} - \frac{1}{\zeta}\frac{dy}{d\zeta} \pm y = 0, \tag{d}$$

the solution of which is known to be expressible in terms of Bessel functions.† Equation (d) with the last term positive has the solution

$$y = \zeta[AJ_1(\zeta) + BN_1(\zeta)],$$

where $J_1(\zeta)$ and $N_1(\zeta)$ denote Bessel functions of the first and second kind respectively, both being of the first order. Equation (d) with the last term negative yields the solution

$$y = \zeta[CJ_1(i\zeta) + DN_1(i\zeta)].$$

Consequently, the general solution of the original differential equation can be obtained as the sum of these two equations. Since

$$J_1(\zeta) = -\frac{dJ_0(\zeta)}{d\zeta} \qquad \text{and} \qquad N_1(\zeta) = -\frac{dN_0(\zeta)}{d\zeta},$$

the solution can also be expressed in terms of Bessel functions of the zero order. This makes it possible to introduce the Z functions of Schleicher (see equation [g] on p. 103), which will then give the result in terms of the real variable $\xi = 2\lambda\sqrt{x}$ as

$$y_x = \sqrt{x}[C_1Z_1'(\xi) + C_2Z_2'(\xi) + C_3Z_3'(\xi) + C_4Z_4'(\xi)]. \tag{93a}$$

* "Beanspruchung und Formänderung zylindrischer Gefässe mit linear veränderlicher Wandstärke," *Vierteljahrsschrift der naturforschende Gesellschaft in Zürich*, 62 (1917), 153. See also F. Kann, "Kegelförmige Behälterböden," *Forscherarbeiten auf dem Gebiete des Eisenbetons*, Heft 29 (Berlin, 1921).

† E. Jahnke and F. Emde, *Tables of Functions* (3d ed.; Leipzig, 1938), p. 146.

Here y_x is the displacement normal to the generator of the cone; x is the distance from the apex; and the derivatives of the functions Z are to be understood as being with respect to the argument ξ. Successive derivatives of the elastic line y_x with respect to x give general formulas for the angular deflection θ_x, bending moment M_x, and shearing force Q_x, the last two being referred to a unit length of the respective hoop circles. Differentiation of the Z functions can be carried out according to the correlations stated in (80). Thus we have

$$\left.\begin{aligned}\theta_x &= \lambda[C_1 Z_2(\xi) - C_2 Z_1(\xi) + C_3 Z_4(\xi) - C_4 Z_3(\xi)],\\ M_x &= \frac{E\delta^3}{12}\frac{\lambda^2}{\sqrt{x}}[C_1 Z_2'(\xi) - C_2 Z_1'(\xi) + C_3 Z_4'(\xi) - C_4 Z_3'(\xi)],\\ Q_x &= -\frac{E\delta^3\lambda^3}{12\,x}\left\{C_1\left[Z_1(\xi) + \frac{2}{\xi}Z_2'(\xi)\right] + C_2\left[Z_2(\xi) - \frac{2}{\xi}Z_1'(\xi)\right]\right.\\ &\qquad \left. + C_3\left[Z_3(\xi) + \frac{2}{\xi}Z_4'(\xi)\right] + C_4\left[Z_4(\xi) - \frac{2}{\xi}Z_3'(\xi)\right]\right\}.\end{aligned}\right\} \quad (93\text{ b–d})$$

The hoop force N_x will be proportional to the deflection y_x, and, according to equation (a), page 31, its value per unit length of the generator will be

$$N_x = \frac{E\delta}{x \tan\alpha}\, y_x . \quad (93\text{e})$$

In addition to this there will be another normal force, T_x, acting in the direction of the generator, which can be obtained as a component of the shearing force Q_x:

$$T_x = Q_x \tan\alpha. \quad (93\text{f})$$

Equations (93 a–f) give the complete solution for the stress system produced in a thin-walled conical shell by equilibrating forces or moments uniformly distributed around any hoop circle of the shell. On account of the approximations involved in the derivation these formulas yield accurate results only if the argument $\xi = 2\lambda\sqrt{x} > 6$. Thus the Z functions in the equations can be replaced by their asymptotic expressions from pages 104–105. The four constants of integration in the general solution are to be determined from the conditions existing at the edges of the cone, two conditions at each edge always being defined in any particular problem. When the cone is closed at the vertex we have $C_3 = C_4 = 0$, and the two remaining constants can be found from the conditions prescribed along the base circle.

If the conical shell is subjected to symmetrically distributed surface loading, the complete solution can be resolved into a membrane and a bending analysis in the manner exemplified by cylindrical shells. First we assume that the shell acts as a membrane, without any bending resistance, in which case the stresses due to distributed surface forces can be calculated purely from considerations of static equilibrium. As a result of this membrane analysis we shall have displacements and rotations along the edges of the shell, which, in general, will not comply with the boundary requirements imposed by the nature of the problem. In order to satisfy the boundary conditions, uniformly distributed moments and forces will have to be applied at the boundaries, thus producing an axially symmetrical bending deformation of the shell. This second part of the solution, that is, the bending analysis, is the one discussed above. In order to have the complete picture, expressions will now be derived for the membrane forces caused by distributed surface loadings on the shell.

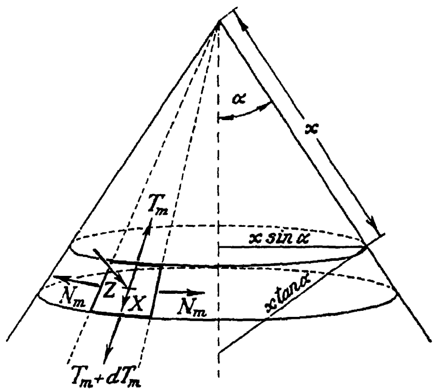

FIG. 99

Let us denote by N_m the circumferential membrane forces and by T_m the radial membrane forces acting per unit length of the respective elements of the shell, and assume that the surface of the shell is loaded with normal forces Z and tangential forces X per unit area, distributed uniformly with respect to the axis of the cone (Fig. 99).

Under such conditions the static equilibrium will require that along any hoop circle at a distance x from the apex we have

$$\frac{d}{dx}(T_m x \cos\alpha) = -x(X\cos\alpha + Z\sin\alpha),$$

which gives

$$T_m = -\frac{1}{x\cos\alpha}\int (X\cos\alpha + Z\sin\alpha)x\,dx. \tag{94}$$

The hoop force N_m can be derived from the equilibrium condition in a direction perpendicular to the surface, which gives directly

$$N_m = -Zx\tan\alpha. \tag{95}$$

With N_m and T_m known over the entire shell, the membrane stress condition is completely defined.

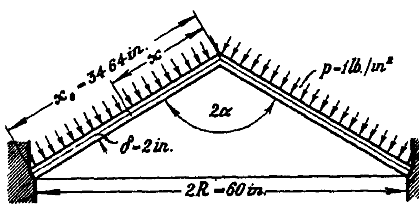

FIG. 100

As an example, let us consider a conical shell of uniform wall thickness $\delta = 2$ in., base-circle radius $R_0 = 30$ in., and full angle of opening $2\alpha = 120°$ subjected to a normal pressure $p = 1$ lb./in.^{2} uniformly distributed over its surface (Fig. 100). The shell is assumed to have supports which prevent horizontal displacements along the base circle, but which introduce no restraint in rotation around the edge. The material of the shell is reinforced concrete, for which we assume a Poisson's ratio value of $\mu = \frac{1}{5}$.

The shell will be considered first as a membrane, when the forces T_m and N_m due to normal loading $Z = p$ lbs./in.^{2} can be obtained from (94) and (95):

$$T_m = -\tfrac{1}{2}px \tan \alpha; \qquad N_m = -px \tan \alpha.$$

The maximum values of these membrane forces occur at $x = x_0 = 34.64$ in., where, because $p = 1$ lb./in.^{2}, we have

$$T_m = -30.00 \text{ lbs./in.} \qquad \text{and} \qquad N_m = -60.00 \text{ lbs./in.}$$

This force system will be accompanied by the following horizontal displacement y_h along the base of the shell:

$$y_h = \frac{R_0}{E\delta}(N_m - \mu T_m) = -\frac{810}{E} \text{ in.}$$

Since the supports of the shell do not permit horizontal displacements along the base circle, the bending forces will have to produce a horizontal displacement $[y_x \cos \alpha]_{x=x_0} = 810/E$, y_x denoting the normal deflection of the shell due to bending, from (93a). In addition to this we shall have another condition, namely, that $[M_x]_{x=x_0} = 0$, which expresses the fact that the rotation of the edge of the shell is not restrained.

By means of (93 a and c) these two boundary conditions can be written as

$$\cos \alpha \sqrt{x_0}[C_1 Z_1'(\xi_0) + C_2 Z_2'(\xi_0)] = \frac{810}{E}$$

and

$$C_1 Z_2'(\xi_0) - C_2 Z_1'(\xi_0) = 0,$$

where

$$\xi_0 = 2\lambda\sqrt{x_0} \quad \text{and} \quad \lambda = \sqrt[4]{\frac{12}{\delta^2 \tan^2 \alpha}}.$$

Substituting here the asymptotic expressions of Z_1' and Z_2' for large values of the argument from page 105, we have

$$C_1 \cos\left(\frac{\xi_0}{\sqrt{2}} + \frac{\pi}{8}\right) - C_2 \sin\left(\frac{\xi_0}{\sqrt{2}} + \frac{\pi}{8}\right) = \frac{810}{E} \frac{1}{\cos \alpha \sqrt{x_0}} \sqrt{2\pi\xi_0} e^{-\xi_0/\sqrt{2}},$$

$$C_1 \sin\left(\frac{\xi_0}{\sqrt{2}} + \frac{\pi}{8}\right) + C_2 \cos\left(\frac{\xi_0}{\sqrt{2}} + \frac{\pi}{8}\right) = 0.$$

The solution of these simultaneous equations gives

$$C_1 = -0.764C \qquad \text{and} \qquad C_2 = -0.645C,$$

where

$$C = \frac{810}{E} \frac{1}{\cos \alpha \sqrt{x_0}} \sqrt{2\pi\xi_0}\, e^{-\xi_0/\sqrt{2}}.$$

Substituting these values together with $C_3 = C_4 = 0$ in (93 a–f), we have the final results of the bending analysis. This gives, for instance, for the bending moment M_x and hoop force N_x, the following expressions:

$$M_x = \frac{810\delta^3\lambda^2}{12x_0 \cos \alpha} \sqrt{\left(\frac{\xi_0}{\xi}\right)^3}\, e^{-(\xi_0-\xi)/\sqrt{2}} \left[0.764 \sin\left(\frac{\xi}{\sqrt{2}} + \frac{\pi}{8}\right) + 0.645 \cos\left(\frac{\xi}{\sqrt{2}} + \frac{\pi}{8}\right)\right],$$

$$N_x = -\frac{810\delta}{x_0 \sin \alpha} \sqrt{\left(\frac{\xi_0}{\xi}\right)^3}\, e^{-(\xi_0-\xi)/\sqrt{2}} \left[0.764 \cos\left(\frac{\xi}{\sqrt{2}} + \frac{\pi}{8}\right) - 0.645 \sin\left(\frac{\xi}{\sqrt{2}} + \frac{\pi}{8}\right)\right].$$

Substituting numerical values in these formulas, we obtain the following values, which are also plotted in Figure 101 for M_x and N_x:

x in.:	34.64	30.00	25.00	20.00	15.00	10.00	5.00;
M_x in. lbs./in.:	0	10.72	10.79	5.79	0.97	−1.09	−0.71;
N_x lbs./in.:	54.00	27.73	6.00	−4.59	−5.64	−2.14	0.63.

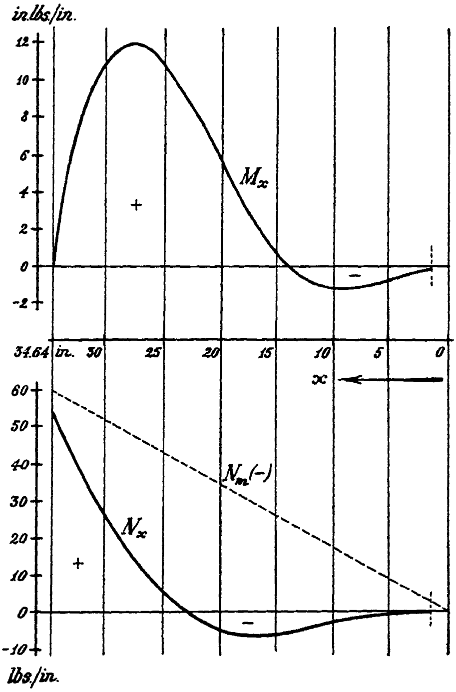

Fig. 101

The final value of the hoop force N will be obtained as the algebraic sum of the quantity N_x given above and the previously derived membrane force N_m. It is seen that along the base circle the hoop force almost vanishes, except for a small value due to the lateral contraction of the material.

For the sake of compaiison, the example above has also been analyzed by means of the existing exact theory for thin conical shells.* The result of this calculation gave for M_x and N_x the following values:

x in.:	34.64	30.00	25.00	20.00	15.00	10.00	5.00;
M_x in. lbs./in.:	0	10.86	10.84	5.79	1.01	−0.98	−0.63;
N_x lbs./in.:	54.00	27.95	6.52	−3.94	−5.15	−2.04	0.40.

The difference between the exact and the approximate values is so small that it cannot be shown in the scale of the diagrams in Figure 101. At the apex of the cone ($x = 0$) both methods give infinite values for M_x and N_x on account of the asymptotic formulas used in the calculation.

* See the references given on page 119.

CHAPTER VI

STRAIGHT BARS UNDER SIMULTANEOUS AXIAL AND TRANSVERSE LOADING

35. *Bars under Axial Tension*

A bar on an elastic foundation will now be assumed to be subjected not only to the vertical loading, but also to a pair of equilibrating horizontal tensile forces N acting in the center of gravity of the end cross sections of the bar.

If we cut out of this bar an infinitely small unloaded element bounded by two verticals a distance dx apart (Fig. 102a), the equilibrium of moments leads to the equation

$$(M + dM) - M + N\,dy - Q_v\,dx = 0$$

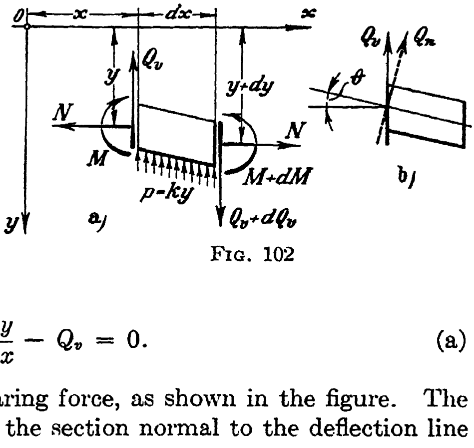

FIG. 102

or

$$\frac{dM}{dx} + N\frac{dy}{dx} - Q_v = 0. \tag{a}$$

Here by Q_v we denote the *vertical* shearing force, as shown in the figure. The *normal* shear Q_n acting in the plane of the section normal to the deflection line can be obtained (see Fig. 102b) as

$$Q_n = Q_v \cos\theta - N \sin\theta,$$

and from this, making the usual assumption that since θ is generally small, $\cos\theta = 1$ and $\sin\theta = \tan\theta = dy/dx$, we get

$$Q_n = Q_v - N\frac{dy}{dx} = \frac{dM}{dx}. \tag{b}$$

In the following derivations we shall use mostly Q_v, but, if need be, we can return from this to Q_n by using (b), where we can also take $dy/dx = \theta$.

Putting $M = -EI(d^2y/dx^2)$ into (a), then differentiating with respect to x, and making the substitution $dQ_v/dx = ky$, we obtain the differential equation of the elastic line for our problem as

$$EI\frac{d^4y}{dx^4} - N\frac{d^2y}{dx^2} + ky = 0. \tag{96}$$

Substituting $y = e^{mx}$, we have the characteristic equation

$$EIm^4 - Nm^2 + k = 0,$$

the four roots of which can be written in the following condensed form:

$$m_{1,2,3,4} = \pm\sqrt{\frac{N}{2EI} \pm i\sqrt{\frac{k}{EI} - \left(\frac{N}{2EI}\right)^2}}. \tag{c}$$

Thus the general solution of (96) is obtained as

$$y = A_1e^{m_1x} + A_2e^{m_2x} + A_3e^{m_3x} + A_4e^{m_4x}. \tag{d}$$

In order to take further steps in the solution we must know whether the expression under the smaller square root in (c) is positive, zero, or negative. According to this we shall distinguish three cases, which will be discussed separately below.

CASE I. $N < 2\sqrt{kEI}$

In most of the practical problems the force N has a value in the region covered by this case.

The expression under the smaller square root in (c) will now be positive, and the four m roots will form two pairs of conjugate complex numbers, which can be put as $m_{1,2,3,4} = \pm(\alpha \pm i\beta)$, where

$$\left.\begin{aligned} \alpha &= \sqrt{\sqrt{\frac{k}{4EI}} + \frac{N}{4EI}} = \sqrt{\lambda^2 + \frac{N}{4EI}}, \\ \beta &= \sqrt{\sqrt{\frac{k}{4EI}} - \frac{N}{4EI}} = \sqrt{\lambda^2 - \frac{N}{4EI}}. \end{aligned}\right\} \tag{97}$$

With this notation the general solution, equation (d), will take the form

$$y = (C_1e^{\alpha x} + C_2e^{-\alpha x}) \cos \beta x + (C_3e^{\alpha x} + C_4e^{-\alpha x}) \sin \beta x. \tag{e}$$

Applying the equation above first to infinitely long bars, we can again conclude, from the trivial assumption that if $x \to \infty$, then $y \to 0$, and $M \to 0$, that C_1 and C_3 must equal zero. The two remaining constants can be determined in any particular case of loading from the conditions of equilibrium and from the symmetry of the elastic line. In this way we obtain for an infinitely long bar loaded at O with a concentrated force P (Fig. 103) the following expressions for the right side of the beam ($x > 0$):

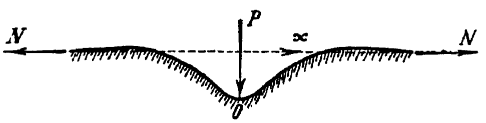

FIG. 103

$$\left.\begin{aligned}
y &= \frac{P}{2k}\frac{\lambda^2}{\alpha\beta}e^{-\alpha x}(\beta\cos\beta x + \alpha\sin\beta x),\\
\theta &= -\frac{P}{4EI}\frac{1}{\alpha\beta}e^{-\alpha x}\sin\beta x,\\
M &= \frac{P}{4}\frac{1}{\alpha\beta}e^{-\alpha x}(\beta\cos\beta x - \alpha\sin\beta x),\\
Q_v &= -\frac{P}{4}\frac{1}{\alpha\beta}e^{-\alpha x}[2\alpha\beta\cos\beta x + (\alpha^2-\beta^2)\sin\beta x].
\end{aligned}\right\}\quad (98\text{ a–d})$$

Under the action of a clockwise moment M_0 at O (Fig. 104) the deflection line and its derivatives for $x > 0$ will be

$$\left.\begin{aligned}
y &= \frac{M_0}{4EI}\frac{1}{\alpha\beta}e^{-\alpha x}\sin\beta x,\\
\theta &= \frac{M_0}{4EI}\frac{1}{\alpha\beta}e^{-\alpha x}(\beta\cos\beta x - \alpha\sin\beta x),\\
M &= \frac{M_0}{4}\frac{1}{\alpha\beta}e^{-\alpha x}[2\alpha\beta\cos\beta x - (\alpha^2-\beta^2)\sin\beta x],\\
Q_v &= -\frac{M_0}{2}\frac{\lambda^2}{\alpha\beta}(\beta\cos\beta x + \alpha\sin\beta x).
\end{aligned}\right\}\quad (99\text{ a–d})$$

From these expressions formulas can be derived by integration for any distributed loading on an infinitely long beam.

It is seen from (98 a–d) and (99 a–d) that, although an axial force N is present, the deflection line and its derivatives remain proportional to the transverse loading P and M_0. The principle of superposition will thus be applicable with regard to transverse loading; consequently, by superposing end-conditioning forces on the infinitely long beam, solutions for finite lengths can be derived in the same manner as was done when no axial force was present (see Chapters II–III). The only difference now will be that in the present case two types of shearing forces are distinguished, Q_n and Q_v, and this distinction will have to be carried through when we are making statements concerning end conditions involving shear.

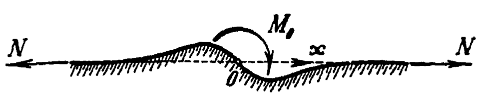

FIG. 104

The conditions for free ends, for instance, will now be

$$M = 0 \quad\text{and}\quad Q_v = \frac{dM}{dx} + N\frac{dy}{dx} = 0.$$

Thus a free end will be produced at any point of an infinite beam by applying such end-conditioning forces as will cancel at that point the bending moment M and the vertical shear Q_v. It is to be noted that, though Q_v will vanish, a normal shear of the value $Q_n = -N(dy/dx) = -N\cdot\theta$ will remain at the free ends.

FIG. 105

Applying in this manner the principle of superposition, we obtain for a semi-infinite bar subjected to a concentrated end force P (Fig. 105) the following expressions:

$$\left.\begin{aligned}
y &= \frac{P}{\beta k}\frac{2\lambda^2}{3\alpha^2-\beta^2}e^{-\alpha x}[2\alpha\beta\cos\beta x+(\alpha^2-\beta^2)\sin\beta x],\\
\theta &= -\frac{P}{EI}\frac{1}{3\alpha^2-\beta^2}\frac{1}{\beta}e^{-\alpha x}(\beta\cos\beta x+\alpha\sin\beta x),\\
M &= -\frac{P}{\beta}\frac{2\lambda^2}{3\alpha^2-\beta^2}e^{-\alpha x}\sin\beta x,\\
Q_v &= -\frac{P}{\beta}\frac{e^{-\alpha x}}{3\alpha^2-\beta^2}[(3\alpha^2-\beta^2)\beta\cos\beta x-(3\beta^2-\alpha^2)\alpha\sin\beta x]\\
Q_n &= Q_v - N\cdot\theta.
\end{aligned}\right\} \quad (100\text{ a–e})$$

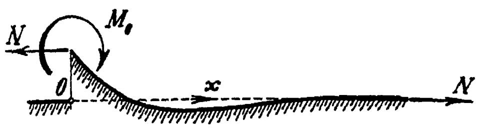

FIG. 106

In a similar way, when a concentrated clockwise moment M_0 is acting at the end of the beam O (Fig. 106) we shall have

$$\left.\begin{aligned}
y &= -\frac{M_0}{EI}\frac{1}{3\alpha^2-\beta^2}\frac{e^{-\alpha x}}{\beta}(\beta\cos\beta x-\alpha\sin\beta x),\\
\theta &= \frac{M_0}{EI}\frac{1}{3\alpha^2-\beta^2}\frac{1}{\beta}e^{-\alpha x}[2\alpha\beta\cos\beta x-(\alpha^2-\beta^2)\sin\beta x],\\
M &= M_0\frac{1}{3\alpha^2-\beta^2}\frac{1}{\beta}e^{-\alpha x}[(3\alpha^2-\beta^2)\beta\cos\beta x-(\alpha^2-3\beta^2)\alpha\sin\beta x],\\
Q_n &= M_0\frac{1}{3\alpha^2-\beta^2}\frac{1}{\beta}e^{-\alpha x}[-4(\alpha^2-\beta^2)\alpha\beta\cos\beta x\\
&\qquad\qquad+(\alpha^4-6\alpha^2\beta^2+\beta^4)\sin\beta x],\\
Q_v &= Q_n + N\cdot\theta.
\end{aligned}\right\} \quad (101\text{ a–e})$$

By the method of superposition explicit formulas can be derived also for beams of finite lengths. This has been carried out for a few elementary loadings, and the results are given below.

a. Free-end Beam with Equal Concentrated End Loads (Fig. 107)

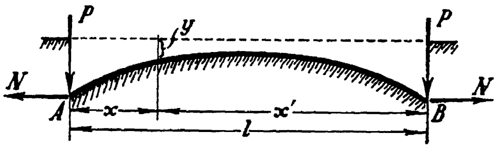

FIG. 107

$$y = \frac{P}{2\lambda^2 EI}\frac{1}{D_1}[2\alpha\beta(\text{Cosh } \alpha x \cos \beta x' + \cos \beta x \text{ Cosh } \alpha x') + (\alpha^2 - \beta^2)(\text{Sinh } \alpha x \sin \beta x' + \sin \beta x \text{ Sinh } \alpha x')], \quad (102)$$

where $D_1 = \beta(3\alpha^2 - \beta^2) \text{ Sinh } \alpha l + \alpha(3\beta^2 - \alpha^2) \sin \beta l.$

b. Free-End Beam with Equal Concentrated End Moments (Fig. 108)

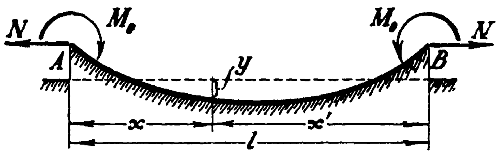

FIG. 108

$$y = \frac{M_0}{EI}\frac{1}{D_1}[\alpha(\text{Cosh } \alpha x \sin \beta x' + \text{Cosh } \alpha x' \sin \beta x) - \beta(\text{Sinh } \alpha x \cos \beta x' + \text{Sinh } \alpha x' \cos \beta x)], \quad (103)$$

where D_1 is as above.

c. Free-End Beam with Concentrated Load at One End (Fig. 109)

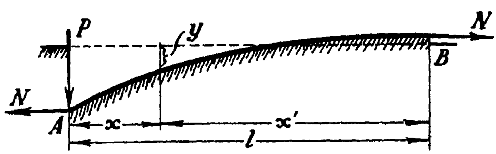

FIG. 109

$$y = \frac{P}{2\lambda^2 EI}\frac{1}{D_1 D_2}\cdot F, \quad \text{where } F, D_1 \text{ and } D_2 \text{ are as shown on next page.} \quad (104)$$

$$F = \{\beta(3\alpha^2 - \beta^2) \text{ Sinh } \alpha l[2\alpha\beta \cos \beta x \text{ Cosh } \alpha x' + (\alpha^2 - \beta^2) \sin \beta x \text{ Sinh } \alpha x']$$
$$- \alpha(3\beta^2 - \alpha^2) \sin \beta l[2\alpha\beta \text{ Cosh } \alpha x \cos \beta x' + (\alpha^2 - \beta^2) \text{ Sinh } \alpha x \sin \beta x']\},$$

D_1 is as before and $D_2 = \beta(3\alpha^2 - \beta^2) \text{ Sinh } \alpha l - \alpha(3\beta^2 - \alpha^2) \sin \beta l.$

d. Free-End Beam with Concentrated Moment at One End (Fig. 110)

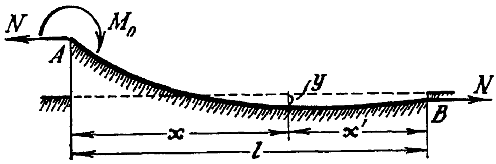

Fig. 110

$$y = \frac{M_0}{EI} \frac{1}{D_1 D_2} [\beta(3\alpha^2 - \beta^2) \text{ Sinh } \alpha l \,(\alpha \text{ Cosh } \alpha x' \sin \beta x - \beta \text{ Sinh } \alpha x' \cos \beta x)$$
$$+ \alpha(3\beta^2 - \alpha^2) \sin \beta l(\beta \text{ Sinh } \alpha x \cos \beta x' - \alpha \text{ Cosh } \alpha x \sin \beta x')], \quad (105)$$

where D_1 and D_2 are as above.

e. Hinged-End Beam with Equal Concentrated End Moments (Fig. 111)

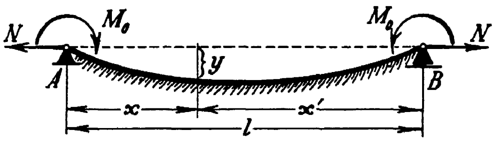

Fig. 111

$$y = \frac{M_0}{EI} \frac{1}{2\alpha\beta} \frac{\text{Sinh } \alpha x' \sin \beta x + \text{Sinh } \alpha x \sin \beta x'}{\text{Cosh } \alpha l + \cos \beta l}. \quad (106)$$

f. Hinged-End Beam with Concentrated Moment at One End (Fig. 112)

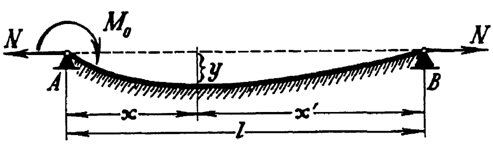

Fig. 112

$$y = \frac{M_0}{EI} \frac{1}{2\alpha\beta} \frac{\text{Cosh } \alpha l \text{ Sinh } \alpha x' \sin \beta x - \cos \beta l \text{ Sinh } \alpha x \sin \beta x'}{\text{Cosh}^2 \alpha l - \cos^2 \beta l} \quad (107)$$

g. Hinged-End Beam under Uniformly Distributed Loading (Fig. 113)

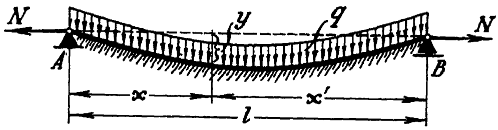

FIG. 113

$$y = \frac{q}{k}\left\{1 - \frac{1}{2\alpha\beta(\text{Cosh } \alpha l + \cos \beta l)}[2\alpha\beta(\text{Cosh } \alpha x \cos \beta x' + \cos \beta x \text{ Cosh } \alpha x') + (\alpha^2 - \beta^2)(\text{Sinh } \alpha x \sin \beta x' + \sin \beta x \text{ Sinh } \alpha x')]\right\}. \quad (108)$$

h. Fixed-End Beam under Uniformly Distributed Loading (Fig. 114)

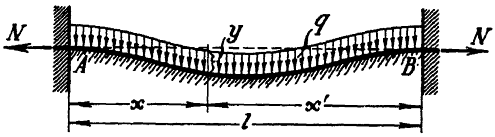

FIG. 114

$$y = \frac{q}{k}\left[1 - \frac{1}{\beta \text{ Sinh } \alpha l + \alpha \sin \beta l}(\alpha \text{ Cosh } \alpha x \sin \beta x' + \beta \text{ Sinh } \alpha x \cos \beta x' + \alpha \sin \beta x \text{ Cosh } \alpha x' + \beta \cos \beta x \text{ Sinh } \alpha x')\right]. \quad (109)$$

The notations in all the formulas above are the same as those introduced previously, namely,

$$\alpha = \sqrt{\sqrt{\frac{k}{4EI}} + \frac{N}{4EI}}, \quad \beta = \sqrt{\sqrt{\frac{k}{4EI}} - \frac{N}{4EI}}, \quad \lambda = \sqrt[4]{\frac{k}{4EI}}$$

where $k = bk_0$, k_0 being the modulus of the foundation and b the width of the beam; EI is the flexural rigidity of the beam, and N is the axial tensile force. In each of the loading cases above expressions for slope, bending moment, or shearing force can be obtained by differentiating the equation of the deflection line with respect to the variable x.

CASE II. $N = 2\sqrt{kEI}$

Thus far the discussion has been limited to Case I, where it was assumed that $N < 2\sqrt{kEI}$. The solutions obtained there can also be applied, after some transformations which will be discussed below, in the two other cases, that is, when N is equal to or larger than $2\sqrt{kEI}$.

Case II can be represented as a limit of Case I by putting $N = 2\sqrt{kEI}$ into the expressions for α and β in (97), by which we get

$$\alpha \to \sqrt{\frac{N}{2EI}} = a \qquad \text{and} \qquad \beta \to 0,$$

By substitution of these values for α and β in any of the formulas previously derived they can be directly transformed for use in the present case. By such a substitution the equation of the deflection line of an infinitely long bar loaded by a concentrated force P, for instance, can be obtained from (98a) as

$$y = \frac{P}{4k} ae^{-ax}(1 + ax), \tag{110}$$

and, similarly, if the same bar is subjected to a concentrated clockwise moment M_0, the equation of the elastic line for $x > 0$ can be obtained from (99a) as

$$y = \frac{M_0 a^3}{4k} e^{-ax} x. \tag{111}$$

It is seen that because of the stretching effect of the axial tensile force N these elastic lines exhibit no more negative ordinates.

CASE III. $N > 2\sqrt{kEI}$

From equation (c), on page 128, it follows that in this case we shall have four real m roots of the characteristic equation. It can be shown that any formula derived in Case I can be directly transformed into Case III by putting instead of β the term $i\bar{\beta}$, in which

$$\bar{\beta} = \sqrt{\frac{N}{4EI} - \sqrt{\frac{k}{4EI}}} = \sqrt{\frac{N}{4EI} - \lambda^2}. \tag{112}$$

This substitution will lead to the transformations

$$\cos i\bar{\beta}x = \text{Cosh}\,\bar{\beta}x, \qquad \text{Cosh}\,i\bar{\beta}x = \cos\bar{\beta}x$$

and

$$\sin i\bar{\beta}x = i\,\text{Sinh}\,\bar{\beta}x, \qquad \text{Sinh}\,i\bar{\beta}x = i\sin\bar{\beta}x.$$

The term α will remain as given by (97). In this manner we get from (98a) for an infinitely long bar loaded at point O with a concentrated force P the deflection line $(x > 0)$:

$$y = \frac{P}{2k}\frac{\lambda^2}{\alpha\bar{\beta}} e^{-\alpha x}(\bar{\beta}\,\text{Cosh}\,\bar{\beta}x + \alpha\,\text{Sinh}\,\bar{\beta}x), \tag{113}$$

and for the case when a concentrated clockwise moment M_0 is acting at O we get, from (99a), for $x > 0$ the expression

$$y = \frac{M_0}{4EI}\frac{1}{\alpha\bar{\beta}} e^{-\alpha x}\,\text{Sinh}\,\bar{\beta}x. \tag{114}$$

36. *Bars under Axial Compression*

The differential equation of bending for the problem when the axial force is compression can be obtained simply by changing the sign of the axial force N in (96), by which we thus get

$$EI\frac{d^4y}{dx^4} + N\frac{d^2y}{dx^2} + ky = 0. \tag{115}$$

The general solution of this equation can be written in the same form as in equation (d) on page 128, but we must distinguish again three cases according to the three different types of roots of the characteristic equation. These cases will be discussed separately below.

CASE I. $N < 2\sqrt{kEI}$

With the same expressions for α and β as those introduced in (97) and used in all the formulas in § 35, that is,

$$\alpha = \sqrt{\sqrt{\frac{k}{4EI}} + \frac{N}{4EI}} = \sqrt{\lambda^2 + \frac{N}{4EI}},$$

$$\beta = \sqrt{\sqrt{\frac{k}{4EI}} - \frac{N}{4EI}} = \sqrt{\lambda^2 - \frac{N}{4EI}},$$

the general solution of the differential equation (115) for the present case can be written as

$$y = (C_1e^{\beta x} + C_2e^{-\beta x}) \cos \alpha x + (C_3e^{\beta x} + C_4e^{-\beta x}) \sin \alpha x. \tag{a}$$

It is seen that this solution for axial compressive force differs from the one for axial tension, equation (e) on page 128, only in that the factors α and β have changed places. By this rule we can obtain formulas for axial compression from those derived in the previous section for axial tension.

Thus we have for an infinitely long bar loaded with a concentrated force P at O and subjected to an axial compressive force N the expression for the deflection line:

$$y = \frac{P}{2k}\frac{\lambda^2}{\alpha\beta}e^{-\beta x}(\alpha \cos \alpha x + \beta \sin \alpha x); \tag{116}$$

and when a positive moment M_0 is applied at O, the deflection line will take the form

$$y = \frac{M_0}{4EI}\frac{e^{-\beta x}}{\alpha\beta} \sin \alpha x. \tag{117}$$

In the same manner, all the formulas given by equations (100a)–(109) can be applied to cases of axial compression by simply interchanging in those equations the symbols α and β.

CASE II. $N = 2\sqrt{kEI}$

If we take an infinitely long bar subjected to a transverse force P and an axial compressive force N belonging to Case II because of its magnitude, the equation of the deflection line can be obtained by interchanging first α and β in (98a), which will give (116), and then putting

$$\alpha \to a = \sqrt{\frac{N}{2EI}} \qquad \text{and} \qquad \beta \to 0$$

This transformation gives

$$y = C \cos ax,$$

where C is found to be an undetermined coefficient of infinite magnitude. From this result and from the fact that the deflection line is represented now by a periodic curve, we can conclude that the limit

$$N = 2\sqrt{kEI}$$

represents that critical value of N under which elastic buckling of the infinitely long bar may occur. Hence at this point we arrived at problems involving elastic stability, which subject is reserved for the next chapter, where it will be discussed in detail.

CASE III. $N > 2\sqrt{kEI}$

Deflection line formulas for values of the axial compressive force N belonging to this case can be obtained from (102)–(109) first by interchanging there α and β, getting thus Case I for compression, and then transforming Case I into Case III by putting instead of β the term $i\bar{\beta}$, in which, as in (112),

$$\bar{\beta} = \sqrt{\frac{N}{4EI} - \lambda^2}.$$

Having obtained in this manner deflection line formulas for bars of finite length we find that for each bar there is a limiting value of the axial compressive force upon the application of which a deflection of the bar can occur without the action of any transverse loading. Such critical values of N for bars of finite length will be determined in Chapter VII, which deals with problems of elastic stability.

37. *Expressions in Terms of Trigonometric Series*

It has been shown in Chapter IV (§§ 24–26) that the deflection line of beams under transverse loading can be readily obtained in the form of trigonometric series in a large variety of loadings and end conditions. The same method can also be applied in the present case when, in addition to transverse loading, the bar is subjected to an axial force N. The effect of this axial force can be taken into account by calculating the work done by this force under the deformation of the bar and adding this work to the potential energy of the whole system (equations [c] and [d] on p. 70). When the loaded bar assumes its deflected position the distance between the ends A and B will change by the amount

$$\Delta l = \frac{1}{2}\int_0^l y'^2\,dx, \tag{a}$$

and consequently the forces N applied at the ends will gain a potential energy of the value

$$V_3 = N\frac{1}{2}\int_0^l y'^2\,dx. \tag{b}$$

Once the loaded beam has found its state of equilibrium, a small change, da_n, from this state will leave unaltered the total energy of the system, which consists of the sum of the potential energy of the loads and the strain energy of deformation. This property has been used in the derivations in § 24, but now, in addition to the previous terms (equation [e] on p. 70), we must consider also the change $(\partial V_3/\partial a_n)\,da_n$ in the potential energy of the axial loading N.

On the basis of an axial tensile force N this additional term has been computed for the main types of loadings discussed in the preceding paragraphs; the final formulas are given below.

A. BAR WITH FREE ENDS SUBJECTED TO A PAIR OF SYMMETRICALLY PLACED P FORCES

The general equation for any a term is not represented now by (58), but takes the form

$$2P\left(\sin\frac{n\pi c}{l} - \frac{2}{n\pi}\right) = a_n\left(n^4\frac{\pi^4 EI}{2l^3} + n^2\frac{\pi^2 N}{2l} + \frac{lk}{2}\right) - \frac{4lk}{\pi^2}\frac{1}{n}\sum_{i=1,3,5,\cdots}^{\infty}\frac{1}{i}a_i. \tag{118}$$

By this formula any number of a terms can be computed; substituting them (59), we obtain an expression for the deflection line.

B. BAR WITH HINGED ENDS

a. Concentrated P Load at Distance c from Left Support

$$y = 2Pl^3\sum_{n=1,2,3,\cdots}^{\infty}\frac{\sin\frac{n\pi c}{l}\sin\frac{n\pi x}{l}}{n^4\pi^4EI + n^2\pi^2l^2N + kl^4}. \tag{119}$$

b. Uniformly Distributed q Loading over the Whole Span

$$y = 4ql^4\sum_{n=1,3,5,\cdots}^{\infty}\frac{\sin\frac{n\pi x}{l}}{n^5\pi^5EI + n^3\pi^3l^2N + n\pi kl^4}. \tag{120}$$

C. BAR WITH FIXED ENDS

a. Two Symmetrically Placed Concentrated P Loads at Distances c from Left and Right Supports Respectively

$$y = 4Pl^3 \sum_{n=1,2,3,\ldots}^{\infty} \frac{\left(1 - \cos\frac{2n\pi c}{l}\right)\left(1 - \cos\frac{2n\pi x}{l}\right)}{n^4 16\pi^4 EI + 4n^2\pi^2 l^2 N + 3kl^4}. \tag{121}$$

b. Uniformly Distributed q Loading over the Whole Span

$$y = 2ql^4 \sum_{n=1,2,3,\ldots}^{\infty} \frac{\left(1 - \cos\frac{2n\pi x}{l}\right)}{n^4 16\pi^4 EI + 4n^2\pi^2 l^2 N + 3kl^4}. \tag{122}$$

In formulas (118)–(122), N has been assumed to be a tensile force; if it is compressive we must reverse its sign in these expressions.

38. *Examples*

1. When computing stresses in a pressure vessel the first step is to separate the head and the cylindrical part and consider each part a membrane which expands on account of the internal pressure, but has no resistance against bending deformation. The membrane expansion will have a different value for each of these parts, and consequently a discontinuity in displacements will result. In order to reassemble the head and the cylinder we must apply at the adjoining edges forces X and moments Y, which will cancel the difference in displacements (and slopes), but will produce at the same time, by bending, the so-called *discontinuity stresses* (discussed in § 27, page 88). Since a meridional strip of the cylinder (see § 11, p. 30), as well as of the head (see § 49, p. 163), can be regarded as a bar on an elastic foundation, the problem can be analyzed on this basis. Inspecting the situation (Fig. 115) more closely, however, we find that for both parts a constant axial force N, the axial resultant of the pressure on the surface of the head, is present. On the basis of the previously derived formulas the effect of this axial force can be taken into account.

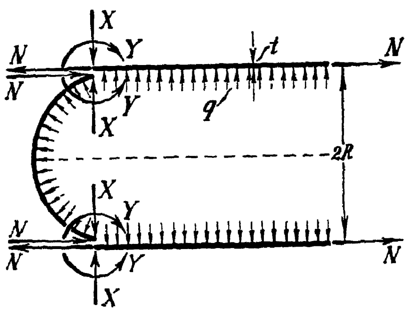

FIG. 115

For a longitudinal element of unit width of the cylinder the value of the axial force will be

$$N = \frac{R^2\pi}{2R\pi} q = \frac{qR}{2},$$

where q denotes the internal pressure lbs./in.^{2} and R is the radius of the cylinder. Substituting this value of N in (97) and putting there $\lambda^2 = \sqrt{3(1-\mu^2)}(1/Rt)$, according to (26), we get for the analysis of the cylindrical part

$$\alpha = \sqrt{\frac{\sqrt{3(1-\mu^2)}}{Rt} + \frac{3Rq}{2Et^3}}, \qquad \beta = \sqrt{\frac{\sqrt{3(1-\mu^2)}}{Rt} - \frac{3Rq}{2Et^3}}.$$

With these α and β values we can proceed to compute the deformation of the cylindrical part due to end forces X and moments Y, using the formulas previously derived in this section. When substituting numerical values for q, R, and t of such magnitude as is at present applied in engineering practice, we find that the second term on the right side of the expressions for α and β is very small in comparison with the first one, which makes the effect of the axial force N on bending stresses negligible. With increasing R and q and decreasing t, however, the second term gains importance. Thus it seems likely that in the future these dimensions may obtain such values that the analysis here outlined will be necessary, especially since the tendency of this axial tensile force N is to stiffen the adjoining parts against deformation and thus increase the discontinuity stresses.

2. Consider a bar supported by an elastic foundation against vertical displacements and, in addition, having its cross sections elastically restrained against rotation. Taking these restraining distributed moments at every point proportional to the slope of the deflection line at that point, $M_c = C(dy/dx)$, from the equilibrium of an infinitesimal element of the bar, we have

$$\frac{dM}{dx} + C\frac{dy}{dx} - Q_v = 0,$$

which gives for the differential equation of the elastic line

$$EI\frac{d^4y}{dx^4} - C\frac{d^2y}{dx^2} + ky = 0.$$

Comparing this equation with (96) we find that the problem above results in the same deflection curve as when the bar is subjected to an axial tensile force $N = C$, which has the dimension of the distributed moment M_c (in. lbs./in. = lbs.).

Problems of this type are found in building or ship construction when a longitudinal girder supported on closely spaced transverse beams may be computed as a bar on elastic foundation. Since there is usually a connection between the longitudinal girder and the transverse beams, the torsional resistance of the transverse beams will oppose the rotation of the corresponding cross section in the longitudinal girder exactly in the manner assumed in the present discussion.*

* The frictional resistance of the foundation can be considered in a similar manner. Such an attempt has been made by A. Francke, "Beitrag zur Berechnung des Eisenbahn-Oberbaues," *Zeitschrift des Architekten- und Ingenieur-Verein zu Hannover* (1894), p. 467. See also Hayashi, *op. cit.* (see p. 5), p. 31.

3. The differential equation of vibration of a rail regarded as a bar on an elastic foundation can be obtained by adding to the statical equation the inertia forces, which will be of the magnitude $(q/g)(d^2y/dt^2)$, the term q denoting the weight of the rail per unit length and g representing the acceleration of gravity.* Thus we have

$$EI\frac{d^4y}{dx^4} + \frac{q}{g}\frac{d^2y}{dt^2} + ky = 0. \tag{a}$$

We can also write

$$\frac{d^2y}{dt^2} = \frac{d^2y}{dx^2}\left(\frac{dx}{dt}\right)^2 + \frac{dy}{dx}\frac{d^2x}{dt^2}. \tag{b}$$

If we consider a concentrated load P moving with a constant velocity v along the rail, we have

$$\frac{dx}{dt} = v \text{ const.} \qquad \text{and} \qquad \frac{d^2x}{dt^2} = 0. \tag{c}$$

Putting these values in (b) and then substituting the result in (a), we obtain

$$EI\frac{d^4y}{dx^4} + \frac{qv^2}{g}\frac{d^2y}{dx^2} + ky = 0, \tag{d}$$

which is the differential equation of the deflection line of a rail under the assumed P force, moving with a constant velocity v. Comparing (d) with (115), we find that the dynamical effect of the moving P force is equivalent to an axial compressive force

$$N = \frac{qv^2}{g}. \tag{e}$$

Investigations show, however, that with the v values used at present this dynamical effect on the deflection of the rail is very small.†

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* The mass of the vibrating foundation is here neglected.

† This problem has been investigated by S. Timoshenko, "Method of Analysis of Statical and Dynamical Stresses in Rail," *Proceedings of the Second International Congress for Applied Mechanics* (Zürich, 1926), pp. 407–418.

The case when the rail is under the action of a P force having constant velocity, but a harmonically varying magnitude (due to the counterweight on the driving wheel), has been discussed by B. Kelsey Hovey, *Beitrag zur Dynamik des geraden Eisenbahngleises* (Dissertation, Göttingen, 1933). See also J. Dörr, "Der unendliche, federnd gebettete Balken unter dem Einfluss einer gleichförmig bewegten Last," *Ingenieur-Archiv*, 14 (1943), 167–192.

CHAPTER VII

ELASTIC STABILITY OF STRAIGHT BARS

39. *General Considerations*

If a straight bar is subjected to purely axial compressive forces of increasing magnitude, at a certain critical value of the compression a sudden lateral deflection (buckling) of the bar will take place. The purpose of the present chapter is to discuss the conditions of buckling for straight bars supported on an elastic foundation and subjected to various types of restraints at the ends. It will be assumed that the elastic foundation surrounds the bar completely, so that whenever the bar deflects laterally a corresponding deformation in the foundation will necessarily be produced.

The critical value of the axial compressive force can be obtained directly from the deflection formulas presented in the previous chapter, where the bar was assumed to be acted upon by simultaneous axial and transverse forces. The numerator of those deflection formulas was always found to be proportional to the transverse loading on the bar (P, M_0 or q), whereas the denominator consisted of terms which were functions of the axial force N only. It is evident that the deflection of the bar may also have a finite value if, while the numerator approaches zero (that is, approaches the condition when no transverse load is acting), the denominator approaches zero too. By setting the denominator thus equal to zero it yields an equation which defines that critical value of the compressive force N_{cr}, under whose sole action, without any transverse loading, lateral deflection or buckling of the bar may occur. This method will be used in the following sections to determine the buckling load of bars with various end conditions and axial load alone. Its application will be shown first for beams of infinite length.

The equation of the deflection line of an infinitely long bar subjected to an axial compressive force N and a transverse force P was obtained in (116) as

$$y = \frac{P}{2k}\frac{\lambda^2}{\alpha\beta}e^{-\beta x}(\alpha \cos \alpha x + \beta \sin \alpha x),$$

where

$$\alpha = \sqrt{\sqrt{\frac{k}{4EI}} + \frac{N}{4EI}}, \qquad \beta = \sqrt{\sqrt{\frac{k}{4EI}} - \frac{N}{4EI}}.$$

According to the method described above, the buckling load N_{cr} will be defined here by the condition that $\alpha\beta = 0$, which, after the values above for α and β

have been substituted, gives

$$\sqrt{\frac{k}{4EI} - \left(\frac{N_{cr}}{4EI}\right)^2} = 0.$$

Hence a bar of unlimited length loaded axially will have the buckling load

$$N_{cr} = 2\sqrt{kEI}. \tag{123}$$

The deflection line of a semi-infinite bar subjected to an axial tensile force N and a concentrated force P at its end was obtained in (100a). This equation can be transformed into one in which the axial force is compression simply by interchanging the terms α and β, getting

$$y = \frac{P}{\alpha k} \frac{2\lambda^2}{3\beta^2 - \alpha^2} e^{-\beta x}[2\alpha\beta \cos \alpha x + (\beta^2 - \alpha^2) \sin \alpha x].$$

The critical value of the axial thrust will thus be defined by the condition

$$3\beta^2 - \alpha^2 = 0.$$

Substituting here the expressions for α and β, we obtain the buckling load for the semi-infinite bar as

$$N_{cr} = \sqrt{kEI}, \tag{124}$$

which is exactly half of the value of N_{cr} obtained above for the infinite bar.

40. *Bars with Free Ends*

As was pointed out previously, any deflection-line formula derived in the preceding chapter under the assumption of an axial tensile force can be applied to a corresponding case where the axial force is compression simply by interchanging the terms α and β throughout. Carrying out this transformation on the free-end-bar formulas, equations (102)–(105), we obtain in the denominator the two types of functions D_1 and D_2, that is,

$$D_{1,2} = \alpha(3\beta^2 - \alpha^2) \text{ Sinh } \beta l \pm \beta(3\alpha^2 - \beta^2) \sin \alpha l, \tag{a}$$

where the plus sign on the right side gives D_1 and the minus sign gives D_2.

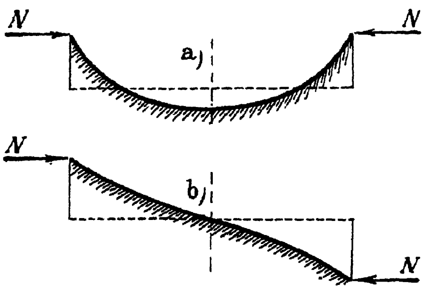

FIG. 116

The function D_1 alone occurs in those deflection formulas, equations (102)–(103), where the elastic line is symmetrical with respect to the center of the bar, indicating that D_2 will correspond to an antisymmetrical deflection curve. According to these two types of deformation, D_1 will furnish the condition for the symmetrical buckling deformation, that is, when the two ends of the bar move out in the same direction (Fig. 116a), whereas D_2 will give the criterion for antisymmetrical buckling (Fig. 116b). The two criteria $D_1 = 0$ and $D_2 = 0$ will yield two values for N_{cr} in every case, that is, for each given ratio of the modulus of the foundation and the flexural rigidity and length of the bar. It is evident that of these two values the smaller one will actually be the critical load, and this will determine, also, the nature of the corresponding buckling deformation (symmetrical or antisymmetrical), as well as the number of waves in the buckling formation.

Putting $D_{1,2} = 0$ in (a), we have

$$\frac{\sin \alpha l}{\operatorname{Sinh} \beta l} = \mp \frac{\alpha(3\beta^2 - \alpha^2)}{\beta(3\alpha^2 - \beta^2)}. \tag{b}$$

Substituting here the expressions for α and β and introducing the new variables

$$x = \frac{N_{cr} l^2}{\pi^2 EI} = \frac{N_{cr}}{N_e} \quad \text{and} \quad y = \sqrt{\frac{kl^4}{EI}}, \tag{c}$$

in which N_e denotes the Euler load for a hinged-end bar of length l and flexural rigidity EI, we can write the buckling conditions of (b) in the form

$$\frac{\sin \frac{1}{2}\sqrt{2y + \pi^2 x}}{\operatorname{Sinh} \frac{1}{2}\sqrt{2y - \pi^2 x}} = \mp \frac{(y - \pi^2 x)\sqrt{2y + \pi^2 x}}{(y + \pi^2 x)\sqrt{2y - \pi^2 x}}, \tag{125}$$

where the minus sign stands for symmetrical and the plus sign for antisymmetrical buckling. Equation (125) can be used for calculating critical loads. For any value of y the corresponding value of x, which is to satisfy the equation, can

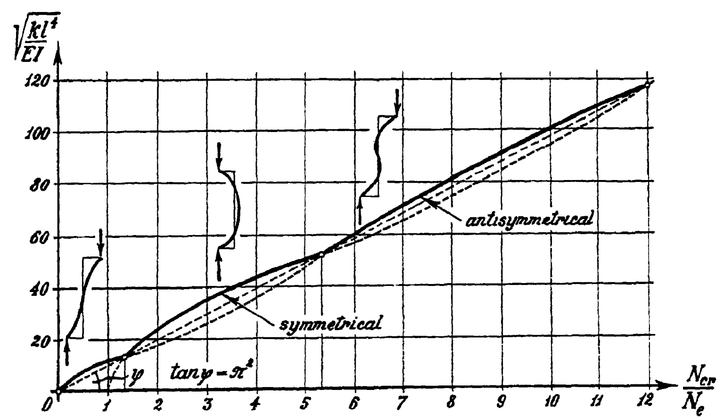

FIG. 117

be determined by a trial-and-error procedure. A graph of the relation between x and y has been calculated from (125) and is shown in Figure 117.* Of the

* Figure 117 and also Figures 118 and 119 were taken from J. Ratzersdorfer, *Die Knickfestigkeit von Stäben und Stabwerken* (Wien, 1936).

two curves in the figure one corresponds to the symmetrical, the other to the antisymmetrical, buckling form, according to the minus and plus signs, respectively, in (125). The curves were drawn alternately with full and dashed lines, the full lines showing in each section the (smaller) critical values of N_{cr} for the corresponding values of y. It is seen that for large values of x and y the ratio y/x approaches π^2, or, in other words, the critical load approaches the value $N_{cr} = \sqrt{kEI}$, which is the buckling load for a semi-infinite beam with a free end. The points where the two curves intersect the line $y = \pi^2 x$ can be determined from (125). Substituting $y = \pi^2 x$, we find that the equality can be satisfied only if $\sin \frac{1}{2}\sqrt{2y + \pi^2 x} = 0$, or $\frac{1}{2}\sqrt{2y + \pi^2 x} = n\pi$ $(n = 1, 2, 3, \cdots)$, which gives for the successive points on the $y = \pi^2 x$ line the following coordinates:

$$x = \tfrac{4}{3}n^2 \qquad \text{and} \qquad y = \tfrac{4}{3}n^2\pi^2, \qquad \text{where } n = 1, 2, 3, \cdots .$$

Thus we find that for values of $y < 4\pi^2/3$ $(n = 1)$ antisymmetrical buckling with one nodal point will give the smallest value for N_{cr}. For values $4\pi^2/3 < y < 16\pi^2/3$ the buckling will be of the first symmetrical mode $(n = 2)$, and for still higher values of y, up to $y = 12\pi^2$, the antisymmetrical form of buckling will again be critical, but in this case with three nodal points, as shown in Figure 117.

41. *Bars with Hinged Ends*

The conditions for buckling of a bar of finite length with hinged ends can be obtained from the deflection formula in (107). Interchanging α and β, we can transform that formula from tension, Case I, into compression, Case I, getting thus

$$y = \frac{M_0}{EI}\frac{1}{2\alpha\beta}\frac{\operatorname{Cosh}\beta l \operatorname{Sinh}\beta x' \sin \alpha x - \cos \alpha l \operatorname{Sinh}\beta x \sin \alpha x'}{(\operatorname{Cosh}\beta l + \cos \alpha l)(\operatorname{Cosh}\beta l - \cos \alpha l)}.$$

Here the plus and the minus signs in the denominator correspond to symmetrical and antisymmetrical deformations respectively.

When no end moment is applied, the bar may have finite deflection only if the denominator of the expression above vanishes. This can occur if $\alpha\beta = 0$, which gives $N_{cr} = 2\sqrt{kEI}$, obtained previously as the critical load for an infinite beam. Since this condition does not take into account the finite length of the bar it is not the complete solution; therefore other possibilities of buckling must be considered. From the same denominator another condition is obtained:

$$\operatorname{Cosh}\beta l \pm \cos \alpha l = 0. \tag{a}$$

Since $\operatorname{Cosh}\beta l > 1$ and $\cos \alpha l < 1$ this equation cannot be satisfied by real values of α and β. Hence it is seen that buckling of a hinged-end beam cannot occur in Case I when $N < 2\sqrt{kEI}$, and the true value of the critical load is to be sought in the range defined by Case III, that is, when $N > 2\sqrt{kEI}$. As was pointed out previously, any formula can be transformed from Case I into Case III by putting, instead of β, the term $i\bar{\beta}$, where $\bar{\beta} = \sqrt{(N/4EI) - \sqrt{k/4EI}}$.

Doing this, we get Cosh $i\bar{\beta}l = \cos\bar{\beta}l$, and the conditions for buckling in (a) will accordingly take the form

$$\cos\bar{\beta}l \pm \cos\alpha l = 0, \tag{b}$$

which is satisfied by

$$(\alpha - \bar{\beta})l = n\pi \tag{c}$$

if $n = 1, 3, 5, \cdots$ is taken for the plus sign and $n = 2, 4, 6, \cdots$ is taken for the minus sign in (b), according to symmetrical and antisymmetrical deformations respectively.

Substituting in (c) the expressions for α and $\bar{\beta}$, we obtain the following formula for the critical load:

$$N_{cr} = n^2\frac{\pi^2 EI}{l^2} + \frac{1}{n^2}\frac{kl^2}{\pi^2}. \tag{126}$$

In every case (for given values of k, l, and EI) n should be determined in such a way as to make this expression for N_{cr} a minimum. The condition $dN_{cr}/dn = 0$ gives

$$n = \frac{l}{\pi}\sqrt[4]{\frac{k}{EI}}. \tag{d}$$

Taking for n an integer number which is the nearest to the value determined from the equation above and substituting that number in (126), we get the value of the critical load. In each instance n determines the number of waves in the deflected form of the bar after buckling.

Substituting $x = \dfrac{N_{cr}l^2}{\pi^2 EI} = \dfrac{N_{cr}}{N_e}$ and $y = \sqrt[2]{\dfrac{kl^4}{EI}}$ in (126), we have

$$n^4 - n^2x + \frac{1}{\pi^4}y^2 = 0. \tag{127}$$

For successive values of n this equation gives a family of curves, shown in Figure 118, representing the relation between y and x, that is, between the dimensions of the bar and the modulus of the foundation on the one hand and the buckling load values on the other hand. The curves intersect the x axis at points where $x = n^2$ and have a common tangent $y = (\pi^2/2)x$, the coordinates of the points of tangency being $x = 2n^2$ and $y = \pi^2 n^2$. The coordinates of the points of intersection between two consecutive curves (n and $n + 1$) are obtained from (127) as $x = 2n(n + 1) + 1$ and $y = \pi^2 n(n + 1)$. Thus we find that for values of $y < 2\pi^2$ buckling will occur in a single wave ($n = 1$); for $2\pi^2 < y < 6\pi^2$ the smallest critical load will correspond to an antisymmetrical deflection form ($n = 2$), and its value will be from five to thirteen times larger than the Euler load for the same bar. For $6\pi^2 < y < 12\pi^2$ the corresponding values of x will be $13 < x < 25$, and buckling will again take place in a symmetrical form, with three waves, as shown in Figure 118. It is seen that in any situation the critical load will differ only slightly from the value $N_{cr} = 2\sqrt{kEI}$, which is the buckling load for an infinitely long bar.

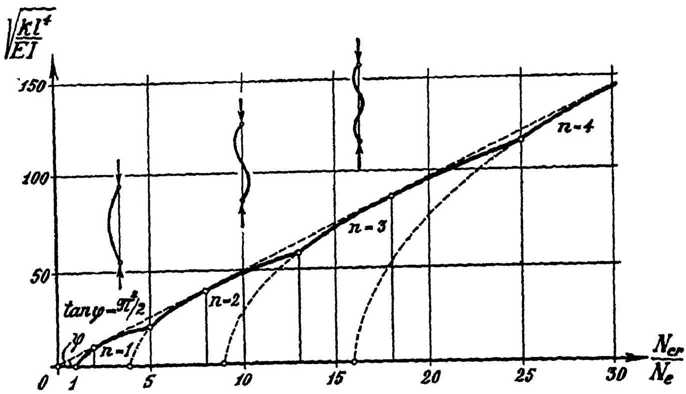

FIG. 118

The criterion of buckling for bars with hinged ends can also be readily obtained from the trigonometric deflection formulas, equations (119)–(120), for beams subject to simultaneous axial tension and transverse loading. Changing the sign of N in these formulas and putting the denominator equal to zero, we have

$$n^4\pi^4EI - n^2\pi^2l^2N_{cr} + kl^4 = 0,$$

which gives the same expression for N_{cr} as that obtained above in (126).

42. *Bars with Fixed Ends*

The condition for symmetrical buckling of a fixed-end bar of finite length can be obtained from (109) after transforming that formula for axial tensile force to one of compression by interchanging the terms α and β. Equating to zero the denominator of the expression so obtained, we have

$$\alpha \operatorname{Sinh} \beta l + \beta \sin \alpha l = 0. \tag{a}$$

By way of comparison with previous cases it is found that the condition for antisymmetrical buckling can be obtained by changing the plus sign in the equation above into a minus sign. The resulting equations

$$\frac{\sin \alpha l}{\operatorname{Sinh} \beta l} = \mp \frac{\alpha}{\beta} \tag{b}$$

cannot, however, be satisfied by real values of β and α, which means that buckling of fixed-end bars will not take place in the region covered by Case I, when $N < 2\sqrt{kEI}$. This criterion of buckling can be transformed from Case I into Case III ($N > 2\sqrt{kEI}$) by putting $\beta = i\bar{\beta}$, which gives $\operatorname{Sinh} i\bar{\beta}l = i \sin \bar{\beta}l$; accordingly, (b) will take the form

$$\frac{\sin \alpha l}{\sin \bar{\beta} l} = \mp \frac{\alpha}{\bar{\beta}}, \tag{c}$$

where

$$\alpha = \sqrt{\frac{N}{4EI} + \sqrt{\frac{k}{4EI}}} \quad \text{and} \quad \bar{\beta} = \sqrt{\frac{N}{4EI} - \sqrt{\frac{k}{4EI}}}.$$

In terms of the new variables

$$x = \frac{N_{cr} l^2}{\pi^2 EI} = \frac{N_{cr}}{N_e} \quad \text{and} \quad y = \sqrt{\frac{kl^4}{EI}},$$

where N_e is the Euler load for a hinged-end bar of length l and flexural rigidity EI, the conditions of buckling in (c) can be written as

$$\frac{\sin \frac{1}{2}\sqrt{\pi^2 x + 2y}}{\sin \frac{1}{2}\sqrt{\pi^2 x - 2y}} = \mp \frac{\sqrt{\pi^2 x + 2y}}{\sqrt{\pi^2 x - 2y}}, \tag{128}$$

where the minus sign on the right side corresponds to symmetrical and the plus sign to antisymmetrical buckling deformations. Assuming various values for y in (128) the corresponding values of x can be calculated by a trial-and-error procedure. In this way we have the pair of curves shown in Figure 119.

These two curves have the same points of intersection with the line $y = (\pi^2/2)(x - 4)$. The coordinates of these points can be calculated by putting $\sin \frac{1}{2}\sqrt{\pi^2 x + 2y} = 0$, which, after the value above for y has been substituted, gives

$$x = 2(n^2 + 1) \quad \text{and} \quad y = \pi^2(n^2 - 1).$$

Thus we find that for values of $y < 3\pi^2$ the critical load will be from four to ten times larger than the Euler load for the same bar with hinged ends, and buckling will occur in a single wave, as shown in Figure 119. For values $3\pi^2 < y < 8\pi^2$

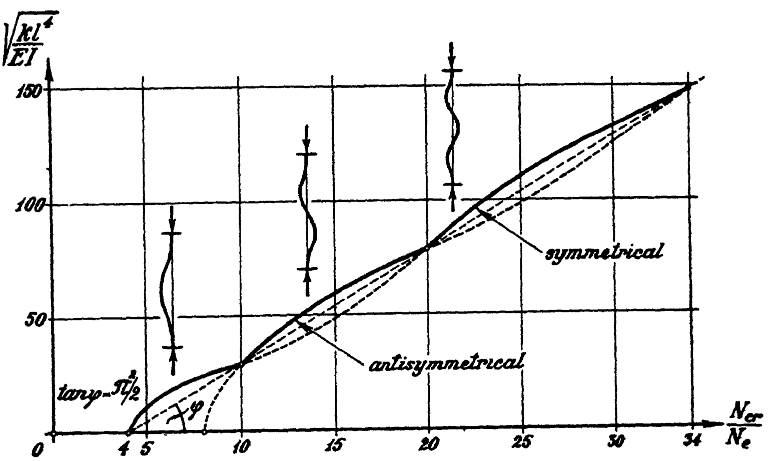

FIG. 119

we get $10 < x < 20$, corresponding to an antisymmetrical buckling form with a nodal point in the middle of the bar; for $8\pi^2 < y < 15\pi^2$ we have $20 < x < 34$, with a three-wave symmetrical deflection form, and so on. It is seen in Figure 119 that the curves always stay close to the straight line $y = (\pi^2/2)(x - 4)$; thus the buckling load for any fixed-end bar with finite length can be taken with good approximation as

$$N_{cr} = 4\frac{\pi^2 EI}{l^2} + 2\sqrt{kEI}.$$

43. *Partially Supported Bars*

A. CANTILEVER BARS

Consider a semi-infinite bar on an elastic foundation with a free unsupported cantilever end, which is loaded at the end point O by an axial force N and a transverse force P, as shown in Figure 120. Since the bending moment at a distance $x < l$ from point O is $M_x = Px + Ny$, the differential equation for the elastic line of the cantilever will be

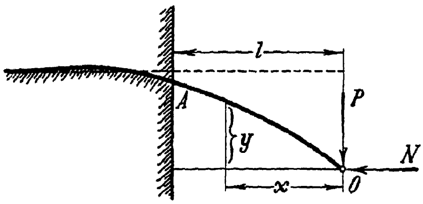

FIG. 120

$$\frac{d^2y}{dx^2} + \frac{N}{EI}y = -\frac{P}{EI}x,$$

and its general solution:

$$y_x = A \sin cx + B \cos cx - \frac{P}{N}x,$$

where

$$c = \sqrt{\frac{N}{EI}}.$$

From the condition that $y = 0$ at $x = 0$ we have $B = 0$; thus the expression for y_x above will take the form

$$y_x = A \sin cx - \frac{P}{N}x. \tag{a}$$

The consecutive derivatives of this equation give

$$\theta_x = Ac \cos cx - \frac{P}{N} \tag{b}$$

and

$$M_x = AN \sin cx. \tag{c}$$

The integration constant A can be determined from the condition that the elastic line must be continuous at point A, or, in other words, that the slope of the elastically supported part (from A to the left) and that of the free cantilever must have the same value at point A.

We have for point A a shearing force $Q = P$ and, from (c), a bending moment $M = AN \sin cl$. Putting these values into (100b) and (101b), we get, after interchanging the terms α and β, the slope at the end A of the semi-infinite bar as

$$\theta_A = \frac{1}{EI}\frac{1}{3\beta^2 - \alpha^2}(P + AN2\beta \sin cl), \tag{d}$$

which must have the same value as the slope at A, obtained from (b) as

$$\theta_A = Ac \cos cl - \frac{P}{N}. \tag{e}$$

Equating expressions (d) and (e), we can determine the value of the integration constant, having thus

$$A = \frac{P\left[\frac{EI}{N}(3\beta^2 - \alpha^2) + 1\right]}{EI(3\beta^2 - \alpha^2)c \cos cl - N2\beta \sin cl}. \tag{f}$$

Substituting this expression for A in equation (a), we find that the cantilever bar may have finite deflection without any transverse loading if, while $P \to 0$, the denominator of A approaches zero also. Thus the condition for buckling is obtained as

$$EI(3\beta^2 - \alpha^2)c \cos cl - N2\beta \sin cl = 0,$$

which can also be written as

$$\frac{3\beta^2 - \alpha^2}{2\beta} = c \tan cl, \tag{g}$$

where α and β denote the same quantities as in the previous sections. Substituting

$$x = N_{cr}\frac{4l^2}{\pi^2 EI} = \frac{N_{cr}}{N_e} \quad \text{and} \quad y = \sqrt{\frac{kl^4}{EI}}$$

in (g), denoting this time by N_e the Euler buckling load for a cantilever bar of length l and flexural rigidity EI, we have

$$y = \frac{\pi^2}{4}\frac{x}{1 - \sin\frac{\pi}{2}\sqrt{x}}, \tag{129}$$

from which for any assumed value of x the corresponding value of y can be readily computed.*

---

* This formula can be used for calculating buckling loads for piles. Investigations of this nature, analytical and experimental, have been made by Hjalmar Grandholm, *On the Elastic Stability of Piles Surrounded by a Supporting Medium,* Ingeniörs Vetenskaps Akademien, Handlingar No. 89 (Stockholm, 1929).

B. BARS WITH FREE MIDDLE SPAN

Consider an infinitely long bar with a free unsupported span A–B loaded in the middle with a concentrated force P, while the bar is subjected also to an axial compressive force N (Fig. 121). The elastic line of this bar can be determined by assuming frictionless hinges at points A and B and thereby resolving the original system into three parts, a simple hinged-end beam in the middle joining two elastically supported semi-infinite beams, whose deformations can be calculated independently from formulas (100 a–b) and (101 a–b). The resulting

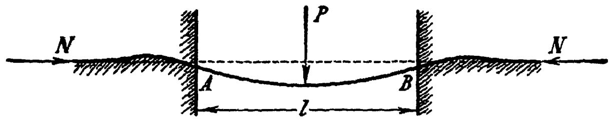

FIG. 121

deflection line will show discontinuities in slope at A and B, and it can be demonstrated by the usual method of computing indeterminate structures that the continuity of the elastic line can be reëstablished by applying at A and B bending moments of the following value:

$$M_{A,B} = \frac{P}{2} \cdot \frac{\dfrac{1}{3\beta^2 - \alpha^2} - \left(\dfrac{l}{2u}\right)^2 \dfrac{1 - \cos u}{\cos u}}{\dfrac{2\beta}{3\beta^2 - \alpha^2} + \dfrac{l}{2u} \tan u}, \tag{h}$$

where α and β denote the quantities given previously and

$$u = \frac{l}{2}\sqrt{\frac{N}{EI}}.$$

Lateral buckling of the bar will occur under the sole action of the axial compressive force N if the latter takes such a value that the denominator* in (h) vanishes, that is, when

$$\frac{2\beta}{3\beta^2 - \alpha^2} + \frac{l}{2u} \tan u = 0. \tag{i}$$

Putting

$$x = \frac{N_{cr} l^2}{\pi^2 EI} = \frac{N_{cr}}{N_e} \quad \text{and} \quad y = \sqrt{\frac{kl^4}{EI}},$$

we can write the condition for buckling in equation (i) as

$$y = \frac{\pi^2 x}{1 + \cos \dfrac{\pi}{2}\sqrt{x}}. \tag{130}$$

For any given value of y the corresponding value of x, which is to determine the buckling load, can be obtained from (130) by successive approximation.

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* Expressions for M and y have the same denominator.

CHAPTER VIII
TORSION OF BARS

44. *Bars of Unlimited Length*

Assume a prismatic bar supported on an elastic foundation and subjected to a concentrated torque T_0 at section O. The problem is to establish expressions for the variation of the twisting moment T and angle of twist φ along the length of the bar. It will be assumed that the foundation can resist tensile and compressive forces equally well and that at any point where the cross section of the bar is twisted through an angle φ a corresponding deformation is produced in the foundation.

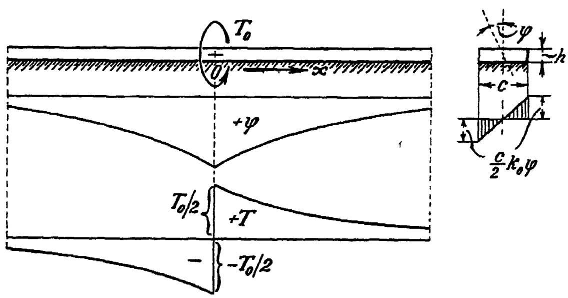

FIG. 122

Ascribing the positive sign to the torque whose vector points in the $+\ x$ direction (Fig. 122) and assuming that the direction of the positive angle of twist coincides with that of the positive torque, we can write the fundamental formula of torsion as

$$T = -GJ\,\frac{d\varphi}{dx}, \tag{a}$$

where GJ is the torsional rigidity of the bar, G being the modulus of rigidity for its material and J a function of its cross-sectional dimensions.* Since the

* When the cross section is rectangular (width c and depth h) we have from Grashof's approximate formula

$$J = \frac{c^3h^3}{3.6(c^2 + h^2)}.$$

For any shape of cross section Saint Venant's formula can be used with good approximation. This gives

$$J = \frac{A^4}{40I_p},$$

where A is the cross-sectional area of the bar and I_p the polar moment of inertia of the cross section.

change in twisting moment will be due to the elastic resistance of the foundation we have

$$\frac{dT}{dx} = -\frac{k_0 c^3}{12} \varphi, \tag{b}$$

where k_0 is the modulus of the foundation in lbs./in.^{3} and c is the width of the bar.

Eliminating T from (a) and (b), we have

$$GJ \frac{d^2 \varphi}{dx^2} - \frac{k_0 c^3}{12} \varphi = 0, \tag{131}$$

the general solution of which is

$$\varphi = Ae^{\alpha x} + Be^{-\alpha x}, \tag{132}$$

where

$$\alpha = \sqrt{\frac{k_0 c^3}{12JG}}.$$

For an infinitely long bar, from the condition that if $x \to \infty$ then $\varphi \to 0$, we have $A = 0$, whereas B can be determined from the equilibrium condition

$$T_0 = 2 \frac{k_0 c^3}{12} \int_0^\infty \varphi \, dx, \tag{c}$$

which gives for the integration constant B the value

$$B = \frac{6\alpha}{k_0 c^3} T_0. \tag{d}$$

Thus, for an infinitely long bar subjected to a torque T_0 at $x = 0$ the angle of twist φ for $x > 0$ values is obtained as

$$\varphi = T_0 \frac{6\alpha}{k_0 c^3} e^{-\alpha x}. \tag{133}$$

Substituting this expression for φ in (a), we get the formula for the twisting moment T at any point where $x > 0$ as

$$T = \tfrac{1}{2} T_0 e^{-\alpha x}. \tag{134}$$

It is seen from the equations above that in an infinitely long bar loaded with a concentrated torque at $x = 0$ the values of φ and T decrease asymptotically as $x \to \infty$ and do not change signs anywhere in the region $0 < x < \infty$.

45. *Bars of Finite Length*

Solutions for bars of finite length can be derived from (133) and (134) by means of superposition. It is easily seen that the condition of a free end ($T = 0$) can be produced in any section of the infinitely long bar by the application of an *end-conditioning torque* which will cancel the twisting moment produced in that section by the given loading. If, instead of a free end, a built-in end is required, the end-conditioning torque has to be determined in such a way as to cancel the angle of twist φ produced by the loading in that section of the infinitely long bar. In this manner, by the application of one end-conditioning torque, a semi-infinite bar can be obtained; in order to produce a bar of finite length, however, two end-conditioning torques are necessary, and in that case the counteraction of these torques on each other also has to be taken into consideration.

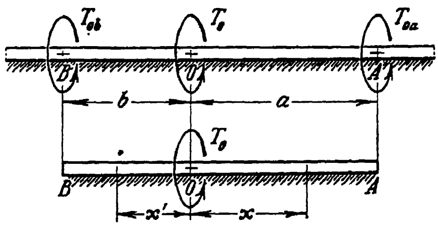

FIG. 123

As an example of this latter case let us derive by the method of superposition the solution for a finite bar of length l with two free ends subjected to a concentrated torque T_0. With the end points A and B at distances a and b respectively from the point of loading (Fig. 123), the end-conditioning torques T_{0a} and T_{0b} which are to produce free ends at A and B have to satisfy the following simultaneous equations:

$$T_0e^{-\alpha a} - T_{0a} + T_{0b}e^{-\alpha l} = 0,$$
$$-T_0e^{-\alpha b} - T_{0a}e^{-\alpha l} + T_{0b} = 0.$$

The solution of these equations gives

$$T_{0a} = T_0\frac{\text{Cosh } \alpha b}{\text{Sinh } \alpha l} \quad \text{and} \quad T_{0b} = T_0\frac{\text{Cosh } \alpha a}{\text{Sinh } \alpha l}.$$

Applying these end-conditioning torques to the right and left of points A and B respectively on the infinitely long bar loaded by T_0, we obtain the case of a bar of finite length with free ends at A and B. The twisting moment at a positive distance x from the point of loading (Fig. 123) will then be

$$T_x = \tfrac{1}{2}T_0e^{-\alpha x} + \tfrac{1}{2}T_{0b}e^{-\alpha(b+x)} - \tfrac{1}{2}T_{0a}e^{-\alpha(a-x)}.$$

Substituting here the values obtained above for T_{0a} and T_{0b}, we have

$$T_x = T_0\frac{\text{Cosh } \alpha b \text{ Sinh } \alpha(a - x)}{\text{Sinh } \alpha l}. \tag{135'}$$

By interchanging a and b in this formula, we obtain an expression for the twisting moment in the O–B region of the bar, at a distance x' from the point of application of the loading torque, as

$$T_{x'} = -T_0\frac{\text{Cosh } \alpha a \text{ Sinh } \alpha(b - x')}{\text{Sinh } \alpha l}. \tag{135''}$$

The equations above are now differentiated with respect to the variables x and x', according to equation (b) on page 152, and the following expressions are obtained for the angles of twist φ_x and $\varphi_{x'}$ in the respective portions, O–A and O–B, of the bar:

$$\varphi_x = T_0 \frac{12\alpha}{k_0 c^3} \frac{\text{Cosh}\,\alpha b \,\text{Cosh}\,\alpha(a - x)}{\text{Sinh}\,\alpha l}, \tag{136'}$$

$$\varphi_{x'} = T_0 \frac{12\alpha}{k_0 c^3} \frac{\text{Cosh}\,\alpha a \,\text{Cosh}\,\alpha(b - x')}{\text{Sinh}\,\alpha l}. \tag{136''}$$

46. *Torsion of Rails*

The torsion problem of a rail with free ends, subjected only to two equal and opposite twisting moments, offers no particular difficulty and can be solved by the general torsional theory. If the ends of the rail are fixed, however, while a concentrated loading torque $2T_0$ is applied at the middle of the rail, the problem is somewhat more complicated and involves, in addition to the twisting of the cross section, a bending of the head and the base sections of the rail.* Thus the torque T_0 transmitted to one half of the rail will be taken up partly by simple torsion T_1 and partly by bending action T_2. The first of these torque components, T_1, will simply be proportional to the rate of change of the angle of twist φ:

$$T_1 = -C \frac{d\varphi}{dx}, \tag{a}$$

where $C = GJ$ represents the torsional rigidity of the rail and J can be calculated by the approximate formula of Saint Venant, given in the note on page 151.

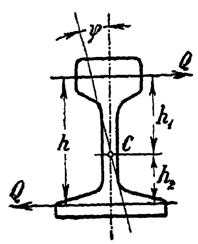

FIG. 124

The second torque component, T_2, will be equivalent to a moment Qh, where Q is the shearing force due to bending of the head and base of the rail and h is the distance between the centroids of the head and base sections (Fig. 124). Neglecting bending of the web, we can determine the position of the center of twist C, around which the cross section will rotate, by the distances

$$h_1 = \frac{hI_2}{I_1 + I_2} \quad \text{and} \quad h_2 = \frac{hI_1}{I_1 + I_2}, \tag{b}$$

where I_1 and I_2 denote the moments of inertia of the head and the base sections respectively with regard to the axis of symmetry of the cross section.

An expression for the shearing force Q can be determined from the equation of bending for the head section of the rail as follows:

$$Q = EI_1 \frac{d^3 y}{dx^3} = EI_1 h_1 \frac{d^3 \varphi}{dx^3}. \tag{c}$$

* The solution of this problem is due to S. Timoshenko, *op. cit.* (see p. 140). See also the paper by Timoshenko and Langer, *op. cit.* (see p. 28).

Substituting here the value obtained above for h_1, we have for the torque T_2 resulting from bending

$$T_2 = Qh = Eh^2 \frac{I_1 I_2}{I_1 + I_2} \frac{d^3\varphi}{dx^3}. \tag{d}$$

According to the requirement that the components T_1 and T_2 keep equilibrium with the loading torque T_0, we have the resulting differential equation of the problem:

$$T_0 = T_1 + T_2 = -C \frac{d\varphi}{dx} + Dh^2 \frac{d^3\varphi}{dx^3}, \tag{137}$$

where

$$D = \frac{EI_1 I_2}{I_1 + I_2}.$$

The solution of this differential equation for a rail of unlimited length is

$$\frac{d\varphi}{dx} = -\frac{T_0}{C}(1 - e^{-\gamma x}), \tag{138}$$

where

$$\gamma = \sqrt{\frac{C}{Dh^2}}.$$

This solution shows that with an increase of the distance x from the point of application of the loading torque, the second term in the parenthesis will lose its significance and the equation will then approach the form obtained for simple torsion (equation [a] on p. 151). The bending of the head and of the base has thus only a localized effect on the torsion of the rail.

Let us assume now that the rail, while subjected to twisting under the conditions described above, is also supported along its entire length by a continuous elastic foundation. The elastic resistance of this foundation, causing the change in torque along the rail, will be proportional at every cross section to the angle of twist of the rail at that section. Thus, denoting this proportionality factor by k_1, we can write

$$\frac{dT}{dx} = -k_1\varphi = -C \frac{d^2\varphi}{dx^3} + Dh^2 \frac{d^4\varphi}{dx^4}, \tag{e}$$

the right side of which was obtained by differentiating (137). Hence we have the differential equation of torsion of a rail supported on an elastic foundation:

$$Dh^2 \frac{d^4\varphi}{dx^4} - C \frac{d^2\varphi}{dx^2} + k_1\varphi = 0. \tag{139}$$

This equation is of the same type as that derived in (96) for beams subjected to simultaneous lateral bending and axial tensile forces, the solution of which was discussed in detail in § 35.

CHAPTER IX
CIRCULAR ARCHES

47. *General Solution of the Elastic Line*

Let us assume that a prismatic beam whose neutral axis in the undeflected state forms an arc of a circle of radius r is subjected to bending forces acting in the plane of curvature of the beam (Fig. 125a). We shall assume that the reaction forces in the foundation will be normal to the axis of the beam and proportional at every point to the radial deflection y of the beam at that point, that is,

$$p = k_0 by = ky,$$

where p denotes the reaction of the foundation per unit length of the beam, b the width of the beam, and k_0 the modulus of the foundation in lbs./in.^{3}

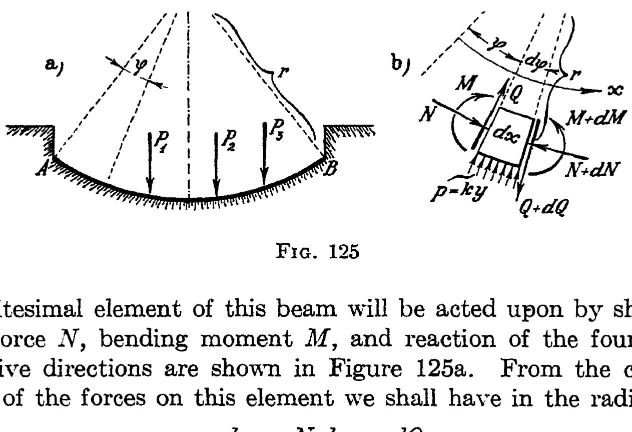

FIG. 125

An infinitesimal element of this beam will be acted upon by shearing force Q, normal force N, bending moment M, and reaction of the foundation pdx, whose positive directions are shown in Figure 125a. From the conditions of equilibrium of the forces on this element we shall have in the radial direction,

$$p\,dx - N\,d\varphi = dQ;$$

in the tangential direction,

$$Q\,d\varphi = dN;$$

while the moment equilibrium condition will require that

$$dM = Qr\,d\varphi.$$

Putting $p = ky$ and $r\,d\varphi = dx$ into the equations above, we have

$$ky - \frac{N}{r} = \frac{dQ}{dx} = \frac{d^2M}{dx^2} \tag{a}$$

and

$$\frac{Q}{r} = \frac{dN}{dx}, \tag{b}$$

which give, after elimination of N and Q,

$$k\frac{dy}{dx} - \frac{1}{r^2}\frac{dM}{dx} = \frac{d^3 M}{dx^3}. \tag{c}$$

At this point we make use of the known differential equation of bending of a circular arch of radius of curvature r and flexural rigidity EI, which, by neglecting the axial deformation due to the normal force N_1, we can write in the form

$$EI\left(\frac{d^2 y}{dx^2} + \frac{y}{r^2}\right) = -M. \tag{d}$$

Differentiating this equation three times with respect to the variable x and equating the results to the left-hand side of (c), we have

$$\frac{d^5 y}{dx^5} + \frac{2}{r^2}\frac{d^3 y}{dx^3} + \left(\frac{k}{EI} + \frac{1}{r^4}\right)\frac{dy}{dx} = 0, \tag{e}$$

which is the fundamental differential equation of bending of circular arches supported on an elastic foundation.

In terms of the variable $\varphi = x/r$ this equation can also be written in the form

$$\frac{d^5 y}{d\varphi^5} + 2\frac{d^3 y}{d\varphi^3} + \eta^2\frac{dy}{d\varphi} = 0, \tag{140}$$

where

$$\eta = \sqrt{\frac{r^4 k}{EI} + 1}.$$

Equation (140) can be solved by the substitution of e^{mx} for y, which gives the characteristic equation

$$m^5 + 2m^3 + \eta^2 m = 0,$$

the five distinct roots of which are

$$m_1 = 0 \qquad \text{and} \qquad m_{2,3,4,5} = \pm(\alpha \pm \beta i),$$

where

$$\alpha = \sqrt{\frac{\eta - 1}{2}} \qquad \text{and} \qquad \beta = \sqrt{\frac{\eta + 1}{2}}. \tag{141}$$

With these notations the general solution of (140) is thus obtained as

$$y = C_0 + (C_1 \operatorname{Cosh} \alpha\varphi + C_2 \operatorname{Sinh} \alpha\varphi) \cos \beta\varphi + (C_3 \operatorname{Cosh} \alpha\varphi + C_4 \operatorname{Sinh} \alpha\varphi) \sin \beta\varphi. \tag{142a}$$

Substituting in (a)–(d) the respective derivatives of the foregoing equation of the elastic line, we have the following expressions for the bending moment and the shearing and normal forces in the beam:

$$\left.\begin{aligned}
M &= -\frac{EI}{r^2}\{C_0 - 2\alpha\beta[(C_1 \operatorname{Sinh}\alpha\varphi + C_2 \operatorname{Cosh}\alpha\varphi)\sin\beta\varphi \\
&\qquad - (C_3 \operatorname{Sinh}\alpha\varphi + C_4 \operatorname{Cosh}\alpha\varphi)\cos\beta\varphi]\}, \\
Q &= 2\alpha\beta\frac{EI}{r^3}[(\alpha C_1 + \beta C_4)\operatorname{Cosh}\alpha\varphi \sin\beta\varphi \\
&\qquad + (\beta C_1 - \alpha C_4)\operatorname{Sinh}\alpha\varphi\cos\beta\varphi + (\alpha C_2 + \beta C_3)\operatorname{Sinh}\alpha\varphi\sin\beta\varphi \\
&\qquad + (\beta C_2 - \alpha C_3)\operatorname{Cosh}\alpha\varphi\cos\beta\varphi], \\
N &= rkC_0 + 2\alpha\beta\frac{EI}{r^3}[(C_1 \operatorname{Sinh}\alpha\varphi + C_2 \operatorname{Cosh}\alpha\varphi)\sin\beta\varphi \\
&\qquad - (C_3 \operatorname{Sinh}\alpha\varphi + C_4 \operatorname{Cosh}\alpha\varphi)\cos\beta\varphi].
\end{aligned}\right\} \quad (142\text{ b–d})$$

The angular deflection θ due to bending can be obtained as the integral of the elementary rotations M/EI along the arch, starting from a suitably chosen origin ($x = 0$ or $\varphi = 0$), where the value of $\theta = \theta_0$ is known:

$$\theta = \int_0^x \frac{M}{EI}\,dx + \theta_0 = \frac{r}{EI}\int_0^\varphi M\,d\varphi + \theta_0 . \tag{f}$$

By the aid of the expression for θ so obtained the horizontal and vertical displacement components, u and v respectively, can also be determined at any point of the arch. If the angle φ is measured from a vertical where $\varphi = 0$, the displacements u and v can be obtained as

$$u = u_0 - \int_0^\varphi \theta r \sin\varphi\,d\varphi \tag{g}$$

and

$$v = v_0 - \int_0^\varphi \theta r \cos\varphi\,d\varphi, \tag{h}$$

where u_0 and v_0 denote the values of these displacements at the origin $\varphi = 0$. It can easily be shown that the components u and v are connected with the radial deflection y in the following manner:

$$u \sin\varphi + v\cos\varphi = y. \tag{i}$$

The integration constants which occur in the general expressions above can be determined in each separate problem from the prescribed end conditions of the arch. Some of these integration constants are often defined in advance by the known character of the deflection line. Taking, for instance, the most frequent case—a symmetrical arch subjected to a symmetrically distributed loading—

if the axis of symmetry is assumed to be vertical and the variable angle φ is measured from this axis, we shall have for the center section ($\varphi = 0$) of the arch

$$\left[\frac{dy}{d\varphi}\right]_{\varphi=0} = 0 \qquad \text{and} \qquad [Q]_{\varphi=0} = 0.$$

These conditions, when substituted in (142 a–c), give

$$\alpha C_2 + \beta C_3 = 0 \qquad \text{and} \qquad \beta C_2 - \alpha C_3 = 0,$$

equations which are satisfied simultaneously only if $C_2 = C_3 = 0$. The conclusion is, therefore, that for any symmetrical type of deformation the integration constants C_2 and C_3 will vanish.

Further simplifications can be obtained in symmetrical cases from knowing that at $\varphi = 0$ we shall then have $\theta_0 = 0$ and $u_0 = 0$. Thus, carrying out the integrations assigned in equations (f) and (g), we obtain for the angular deflection θ and the horizontal displacement component u the following general expressions:

$$\theta = -\frac{\varphi}{r} C_0 + \frac{2\alpha\beta}{\alpha^2 + \beta^2}\frac{1}{r}[(\alpha C_1 - \beta C_4) \operatorname{Cosh} \alpha\varphi \sin \beta\varphi - (\beta C_1 + \alpha C_4) \operatorname{Sinh} \alpha\varphi \cos \beta\varphi], \quad (143)$$

$$u = C_0(\sin \varphi - \varphi \cos \varphi) - \frac{2\alpha\beta}{\alpha^2 + \beta^2}[(\alpha C_1 - \beta C_4)I_1 - (\beta C_1 + \alpha C_4)I_2], \quad (144)$$

where

$$I_1 = \int_0^{\varphi} \operatorname{Cosh} \alpha\varphi \sin \beta\varphi \sin \varphi d\varphi$$
$$= \frac{1}{4\beta}\left\{\operatorname{Sinh} \alpha\varphi \left[\frac{\alpha}{\beta - 1} \cos (\beta - 1)\varphi - \frac{\alpha}{\beta + 1} \cos (\beta + 1)\varphi\right] + \operatorname{Cosh} \alpha\varphi[\sin (\beta - 1)\varphi - \sin (\beta + 1)\varphi]\right\}.$$

$$I_2 = \int_0^{\varphi} \operatorname{Sinh} \alpha\varphi \cos \beta\varphi \sin \varphi \, d\varphi$$
$$= \frac{1}{4\beta}\left\{\operatorname{Cosh} \alpha\varphi \left[\frac{\alpha}{\beta + 1} \sin (\beta + 1)\varphi - \frac{\alpha}{\beta - 1} \sin (\beta - 1)\varphi\right] - \operatorname{Sinh} \alpha\varphi[\cos (\beta + 1)\varphi - \cos (\beta - 1)\varphi]\right\}.$$

48. *Circular Ring*

As an application of the general formulas derived above let us now investigate the case when the curved beam forms a complete ring and is subjected to a concentrated force P at its vertex (Fig. 126). This problem is the same as that of a pipe line surrounded by an elastic foundation and loaded at the top with a line loading of constant intensity along the length of the pipe.*

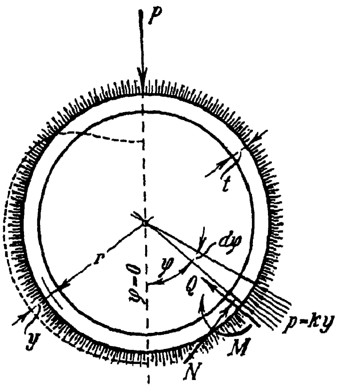

FIG. 126

As in previous problems, the foundation will be assumed to resist tensile and compressive forces equally well. The radial deflection y will be considered positive when producing compression in the foundation, and the positive directions for the forces Q, N, and bending moment M will be those shown in Figure 126. The radius of the center line of the ring will be denoted by r, the thickness by t (the width will be assumed equal to unity), and the bending rigidity by $EI = Et^3/12$; the variable φ in radians will be measured from the base of the symmetry line having for the vertex $\varphi = \pi$.

From a consideration of the symmetry of the problem we find that the deformation and the stress distribution in the ring will have to satisfy the following conditions:

1. $\dfrac{dy}{d\varphi} = 0$ at $\varphi = 0$; 2. $\dfrac{dy}{d\varphi} = 0$ at $\varphi = \pi$;

3. $Q = 0$ at $\varphi = 0$; 4. $Q = -\dfrac{P}{2}$ at $\varphi = \pi$;

$$5.\quad \frac{r}{EI}\int_0^\pi M\,d\varphi = -\frac{1}{r}\int_0^\pi \left(\frac{d^2y}{d\varphi^2} + y\right)d\varphi = 0.$$

The last of these equations expresses the condition that there can be no relative rotation between the base and top sections of the ring.

It is seen that the five conditions above will completely define the values of the five integration constants C when substituted in the general solution in (142 a–d). Making this substitution, we shall obtain the first four of these conditions in the following form:

1. $\alpha C_2 + \beta C_3 = 0;$

2. $C_1(\alpha \text{ Sinh } \alpha\pi \cos \beta\pi - \beta \text{ Cosh } \alpha\pi \sin \beta\pi)$
$$+ C_4(\alpha \text{ Cosh } \alpha\pi \sin \beta\pi + \beta \text{ Sinh } \alpha\pi \cos \beta\pi) = 0;$$

* This problem has been investigated by A. Voellmy, "Erddruck auf elastisch eingebettete Rohre," *International Association for Bridge and Structural Engineering, Publications*, 4 (Zürich, 1936), 591–611.

If the circular ring is regarded as a cross section of a pipe line, its bending rigidity EI should be increased in a ratio $1 : (1 - \mu)$, Poisson's ratio for the material of the pipe being denoted by μ.

3. $\beta C_2 - \alpha C_3 = 0;$

4. $C_1(\alpha \text{ Cosh } \alpha\pi \sin \beta\pi + \beta \text{ Sinh } \alpha\pi \cos \beta\pi)$

$$- C_4(\alpha \text{ Sinh } \alpha\pi \cos \beta\pi - \beta \text{ Cosh } \alpha\pi \sin \beta\pi) = -\frac{P}{4EI}\frac{r^3}{\alpha\beta}.$$

From conditions 1 and 3 we have

$$C_2 = C_3 = 0, \tag{a}$$

whereas the simultaneous solution of conditions 2 and 4 gives the following expressions for the constants C_1 and C_4:

$$\left.\begin{aligned} C_1 &= -\frac{P}{4EI}\frac{r^3}{\alpha\beta}\frac{\alpha \text{ Cosh } \alpha\pi \sin \beta\pi + \beta \text{ Sinh } \alpha\pi \cos \beta\pi}{\eta(\text{Sinh}^2 \alpha\pi + \sin^2 \beta\pi)}, \\ C_4 &= \frac{P}{4EI}\frac{r^3}{\alpha\beta}\frac{\alpha \text{ Sinh } \alpha\pi \cos \beta\pi - \beta \text{ Cosh } \alpha\pi \sin \beta\pi}{\eta(\text{Sinh}^2 \alpha\pi + \sin^2 \beta\pi)}. \end{aligned}\right\} \tag{b}$$

The last of the constants C_0 can be determined from condition 5, which, by means of (142b), can be written in the form

$$\int_0^\pi \left(C_0 \frac{1}{2\alpha\beta} - C_1 \text{ Sinh } \alpha\varphi \sin \beta\varphi + C_4 \text{ Cosh } \alpha\varphi \cos \beta\varphi\right) d\varphi = 0.$$

Substituting here the expressions obtained above for C_1 and C_4 and carrying out the assigned integration, we get

$$C_0 = \frac{P}{2\pi EI}\frac{r^3}{\eta^2}. \tag{c}$$

All the five integration constants having been thus determined, they can be substituted in the general solution expressed by (142 a–d), and will give, then, the final formulas for the deflection and force components along the circumference of the elastically supported ring in Figure 126.

Introducing the notations

$$A = \frac{\alpha \text{ Cosh } \alpha\pi \sin \beta\pi + \beta \text{ Sinh } \alpha\pi \cos \beta\pi}{\eta(\text{Sinh}^2 \alpha\pi + \sin^2 \beta\pi)},$$

$$B = \frac{\alpha \text{ Sinh } \alpha\pi \cos \beta\pi - \beta \text{ Cosh } \alpha\pi \sin \beta\pi}{\eta(\text{Sinh}^2 \alpha\pi + \sin^2 \beta\pi)},$$

where

$$\eta = \sqrt{\frac{r^4 k}{EI} + 1}, \qquad \text{and} \qquad \alpha = \sqrt{\frac{\eta - 1}{2}}, \qquad \beta = \sqrt{\frac{\eta + 1}{2}},$$

we can write the final expressions as

$$\left.\begin{aligned}
y &= \frac{Pr^3}{4\alpha\beta EI}\left(\frac{2\alpha\beta}{\pi\eta^2} - A \operatorname{Cosh} \alpha\varphi \cos \beta\varphi + B \operatorname{Sinh} \alpha\varphi \sin \beta\varphi\right), \\
M &= -\frac{Pr}{2}\left(\frac{1}{\pi\eta^2} + A \operatorname{Sinh} \alpha\varphi \sin \beta\varphi + B \operatorname{Cosh} \alpha\varphi \cos \beta\varphi\right), \\
Q &= -\frac{P}{2}\,[(\alpha A - \beta B) \operatorname{Cosh} \alpha\varphi \sin \beta\varphi \\
&\qquad + (\beta A + \alpha B) \operatorname{Sinh} \alpha\varphi \cos \beta\varphi], \\
N &= \frac{P}{2}\left(\frac{\eta^2 - 1}{\pi\eta^2} - A \operatorname{Sinh} \alpha\varphi \sin \beta\varphi - B \operatorname{Cosh} \alpha\varphi \cos \beta\varphi\right).
\end{aligned}\right\} \quad (145\text{ a–d})$$

For the point of application of the load ($\varphi = \pi$) the formulas above give the following maximum values for deflection and bending moment:

$$\left.\begin{aligned}
[y]_{\varphi=\pi} &= \frac{Pr^3}{4\alpha\beta EI}\left(\frac{2\alpha\beta}{\pi\eta^2} - \frac{\beta \operatorname{Sinh} \alpha\pi \operatorname{Cosh} \alpha\pi + \alpha \sin \beta\pi \cos \beta\pi}{\eta(\operatorname{Sinh}^2 \alpha\pi + \sin^2 \beta\pi)}\right), \\
[M]_{\varphi=\pi} &= -\frac{Pr}{2}\left(\frac{1}{\pi\eta^2} + \frac{\alpha \operatorname{Sinh} \alpha\pi \operatorname{Cosh} \alpha\pi - \beta \sin \beta\pi \cos \beta\pi}{\eta(\operatorname{Sinh}^2 \alpha\pi + \sin^2 \beta\pi)}\right).
\end{aligned}\right\} \quad (146\text{ a–b})$$

It is easily seen that by putting $P = 1$ the equations in (145 a–d) also give the equations of the *influence lines* of y, M, Q, and N for any cross section of the ring. The force $P = 1$ acting at $\varphi = \varphi_0$ will produce, for instance, the same amount of bending moment in the section at $\varphi = \pi$ as the force $P = 1$ acting at $\varphi = \pi$ produces in the section at $\varphi = \varphi_0$. On the basis of this relationship we can determine the force and displacement components at any point on the elastically supported ring due to any combination of radial loading.

In addition to its use in the problem of pipe lines, the elastically supported ring has structural applications also. The assumptions made in our analysis regarding the nature of the elastic foundation are truly realized, for instance, in a ring which is reinforced by a large number of closely spaced radial spokes (Fig. 127a). If initial tension is put into the spokes they are able to resist compressive as well as tensile forces, and the force produced in the spokes by the external loading on the ring will be proportional to the radial deflection of the ring at every point. This sort of structure is used for bulkheads in rigid airships;* a more everyday example occurs in certain types of vehicle wheels† (Fig. 127b).

* See L. H. Donnell, H. B. Gibbons, and E. L. Shaw, "Analysis of Spoked Rings," *Journal of Applied Mechanics, Transactions of the American Society of Mechanical Engineers*, Vol. 63 (1941), Paper A-68.

† See A. J. S. Pippard and W. E. Francis, "On a Theoretical and Experimental Investigation of the Stresses in a Radially Spoked Wire Wheel under Loads Applied to the Rim," *Philosophical Magazine*, Seventh Series, 11 (1931), 233–285.

In these wheels the external loading is balanced by the resultant of the spoke forces, which is transmitted through the axle. The stress analysis of this problem is but a direct application of the formulas obtained above in (145 a–d).

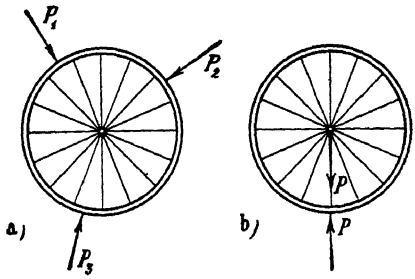

FIG. 127

49. *Spherical Shell*

It has been demonstrated in previous chapters that axially symmetrical deformation of cylindrical and conical shells can be analyzed by the theory of bending of straight beams on an elastic foundation. In a similar manner it will be shown now that the bending analysis of spherical shells can be reduced to the problem of flexure of elastically supported curved beams. The curved beam

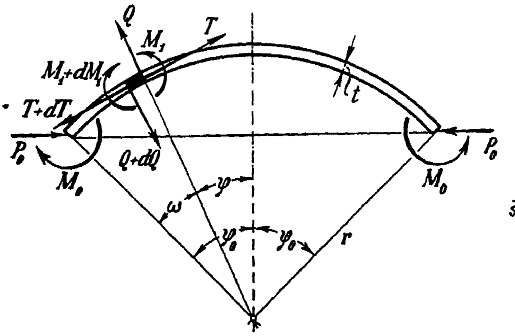

FIG. 128

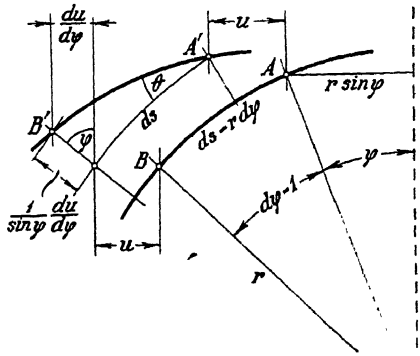

FIG. 129

in this case will be a meridional element of the shell, of variable width since it tapers down to zero at the vertex; the modulus of the elastic foundation, represented by the resilience of the hoop elements in the shell, will also vary on account of the decreasing radii of the hoop circles toward the vertex.

The shell will be assumed to be under the action of an edge loading consisting of horizontal forces P_0 and moments M_0 uniformly distributed along the base circle of the shell (Fig. 128). The deformation produced by such a loading will be symmetrical with respect to the axis of the shell and can be defined in terms of only one independent coordinate, for which purpose the angle φ will be chosen, measured in radians from the axis of the shell.

In order to describe the bending deformation of the shell, let us assume on the undeformed meridional circle two points, A and B, which in terms of the variable φ are spaced an angle $d\varphi = 1$ apart (Fig. 129). These two points, after the deformation of the shell, will be in the positions A' and B', respectively, but the distance between them, $ds = rd\varphi$, will remain unchanged if only the bending deformation is taken into account and the change of the arc length due to normal forces is disregarded.

If we denote the horizontal displacement of point A by u, taking it as positive when accompanied by an increase of the radius of the hoop circle, the horizontal displacement for point B will be $u + du/d\varphi$. By means of these horizontal displacement components it is possible to express the change of slope of the meridional circle θ due to bending. Assuming that θ is small, so that we may put $\tan \theta = \sin \theta = \theta$ and $\cos \theta = 1$, we see from Figure 129 that

$$\theta = \frac{1}{r \sin \varphi} \frac{du}{d\varphi}, \tag{a}$$

its sign being positive if, like the displacement u, it is accompanied by an extension of the corresponding hoop circle.

By means of θ we can obtain the meridional bending moment M_1 as

$$M_1 = -EI \frac{1}{r} \frac{d\theta}{d\varphi}, \tag{b}$$

denoting by EI the flexural rigidity of the meridional beams. Since the symmetry of deformation prevents any lateral (cross-sectional) deformation of these beams, in place of EI we can put in this formula $Et^3/[12(1 - \mu^2)]$, where t is the thickness of the shell and μ is Poisson's ratio for its material. As a result of this lateral restraint, M_1 will always be accompanied by a bending moment acting normally to the meridional beam, that is, in the direction of the hoop circle, and the value of this hoop bending moment will be

$$M_2 = \mu M_1. \tag{c}$$

The bending moments M_1 and M_2 in the equations above will be assumed to act on unit lengths of the respective hoop and meridional circles. Accordingly their dimension will be in. lbs./in. and will be considered positive when producing compression in the outer fibers of the shell.

So far we have expressed both θ and M in terms of the horizontal displacement component u. The next step will be to find a relationship between u and some of the force components in the shell. Since horizontal displacement can take place only if accompanied by a stretching of the corresponding hoop circle we find, in a hoop circle of radius $r \sin \varphi$, the displacement u as a function of the hoop normal force N:

$$u = \frac{r \sin \varphi}{Et} (N - \mu T). \tag{d}$$

The dimension of N in the formula above is lbs./in. and will be assumed to act per unit length of the meridional circle. This force represents the reaction of the elastic foundation against horizontal displacement; it will, therefore, account for the change in shearing force Q of the meridional beam:

$$N = -\frac{dQ}{d\varphi}. \qquad \text{(e)}$$

The shearing force Q can also be expressed in terms of the meridional bending moment M_1 by a consideration of equilibrium of the moments in the plane of the meridional beam (Fig. 128). Taking into account that the width of the beam is proportional to $\sin \varphi$ at every point, we have

$$\frac{d}{d\varphi}(M_1 \sin \varphi) = -Qr \sin \varphi. \qquad \text{(f)}$$

The last of the unknowns in the shell analysis, the meridional normal force T lbs./in. (Fig. 130), can be obtained simply as a component of the shearing force Q:

$$T = -Q \cot \varphi. \qquad \text{(g)}$$

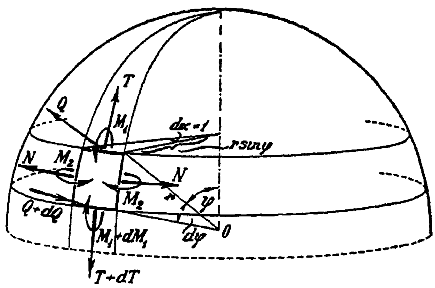

FIG. 130

In the seven equations (a–g) we have all the data necessary to derive expressions for the seven unknown quantities (forces Q, N, T, moments M_1, M_2, and displacement components θ, u) which define the axially symmetrical deformation of a spherical shell. By successive elimination of the unknown quantities from these seven equations the problem can be reduced to the solution of one differential equation containing one unknown quantity. In order to accomplish this let us first substitute in (b) the expression for θ in (a). Carrying out the assigned differentiation, we can write the result as

$$\frac{Et^3}{12(1-\mu^2)r^2}\left(-\frac{d^2u}{d\varphi^2} + \frac{\cos\varphi}{\sin\varphi}\frac{du}{d\varphi}\right) = M_1 \sin \varphi. \qquad \text{(h)}$$

Differentiating the left side of this equation with respect to φ, we find that the result, according to (f), will be equal to $-Qr \sin \varphi$. Substituting then for u its expression in (d), where, according to (e), $N = -dQ/d\varphi$, we find that all derivatives of u can be expressed in terms of those of Q. As a final result of this operation we obtain one differential equation containing only the shearing force Q:

$$\frac{d^4Q}{d\varphi^4} + \frac{2\cos\varphi}{\sin\varphi}\frac{d^3Q}{d\varphi^3} - \frac{1+\sin^2\varphi}{\sin^2\varphi}\frac{d^2Q}{d\varphi^2} + \frac{\cos\varphi}{\sin\varphi}\frac{dQ}{d\varphi} + 12(1-\mu^2)\frac{r^2}{t^2}Q = 0. \qquad (147)$$

A general solution of this equation would give us the variation of Q along the meridional circle. In deriving an approximate solution, however, it is not necessary to take into account all derivatives of Q in the differential equation above. It has been observed in experiments with thin shells that the bending action of edge forces is concentrated mainly in the region around the edge of the shell. Because of this character of the deformation the amplitude of all force and displacement components will decrease rapidly toward the vertex of the shell and their higher derivatives will be of numerically much greater magnitude than the lower ones, which can even be disregarded without much affecting the accuracy of the results. Such an approximation will be made use of in the present case. It can be shown that by putting in (147)

$$Q = \frac{1}{\sqrt{\sin\varphi}}\bar{Q},$$

the terms containing the third derivative of Q will cancel out, and then only the lower derivatives will have to be neglected in order to bring (147) into the simplified form

$$\frac{d^4\bar{Q}}{d\varphi^4} + 4\lambda^4\bar{Q} = 0. \tag{148}$$

where

$$\lambda^4 = 3(1 - \mu^2)\frac{r^2}{t^2}.$$

It is seen that the expression thus obtained is the same as the differential equation of bending of straight beams on an elastic foundation, and its solution, therefore, can be written directly as

$$\bar{Q} = e^{\lambda\varphi}(C_1 \cos\lambda\varphi + C_2 \sin\lambda\varphi) + e^{-\lambda\varphi}(C_3 \cos\lambda\varphi + C_4 \sin\lambda\varphi).$$

Assuming now that we are dealing with a spherical shell which is closed at the top,* as shown in Figure 128, we see that $\bar{Q}$ and its second derivative will have to approach zero in the region of the vertex of the shell, which will require that we put in the equation above

$$C_1 = -C_3 \qquad \text{and} \qquad C_2 = C_4.$$

In determining the two remaining integration constants from the conditions at the base of the shell, it is convenient to introduce in place of φ a new variable ω such that $\varphi = \varphi_0 - \omega$, where φ_0 is the half angle of opening of the shell (Fig. 128). Having done this, we can, after some trigonometrical transformation, write

* If the shell has a circular opening at the top the four integration constants will all have, as a rule, different values, and will have to be determined in such a way as to satisfy simultaneously the conditions prescribed by the nature of the problem for the upper and the lower edges of the shell.

the solution of (148) for the original unknown, the shearing force Q, in the following form:

$$Q = C \frac{e^{-\lambda\omega}}{\sqrt{\sin(\varphi_0 - \omega)}} \sin(\lambda\omega + \psi), \tag{149a}$$

where C and ψ are the constants of integration. This approximate formula for Q is the same as the one obtained in the theory of thin shells by an asymptotic integration of the differential equation of bending.*

* The exact form of the differential equation for Q, derived first by H. Reissner ("Spannungen in Kugelschalen," *Müller-Breslau Festschrift* [Leipzig, 1912], p. 181) from the flexural theory of thin shells, is

$$\frac{d^4Q}{d\varphi^4} + \frac{2\cos\varphi}{\sin\varphi}\frac{d^3Q}{d\varphi^3} - \frac{3 - \sin^2\varphi}{\sin^2\varphi}\frac{d^2Q}{d\varphi^2} + \frac{\cos\varphi(3 + 2\sin^2\varphi)}{\sin^3\varphi}\frac{dQ}{d\varphi} + \left[(1 - \mu^2)\left(1 + 12\frac{r^2}{t^2}\right) - \frac{3}{\sin^4\varphi}\right]Q = 0$$

It has been shown by E. Meissner ("Das Elastizitätsproblem für dünne Schalen,' *Physik*, 14 [1913], 343; also "Über Elastizität und Festigkeit dünner Schalen," *Vierteljahrsschrift der naturforschende Gesellschaft in Zürich*, 60 [1915], 23) that a rigorous solution of this equation can be obtained in the form of hypergeometric series. In this solution, however, the thickness of the shell appears under negative exponents, thus rendering the series slowly convergent for small values of t, which are of particular interest in engineering practice. In order to obtain a more generally applicable solution O. Blumenthal ("Über die asymptotische Integration, . . .," *Zeitschrift für Mathematik und Physik*, 62 [1914], 343–358) was the first to use the method of asymptotic integration. His first approximation, by canceling out the third derivative of Q and then neglecting the lower derivatives, leads to the same formula for Q as the one obtained in (149a). This identity of results is naturally due to the fact that the coefficient of the fourth and third derivatives in the exact differential equation for Q is the same as those derived here from the flexural theory of elastically supported curved beams, equation (147). Using this solution for Q in deriving expressions for the other unknown quantities, however, we shall find that the shell theory will give somewhat different, and more accurate, results than those obtained in (149 b–g) by the beam theory, since the latter disregards the effect of some of the stress components present in a thin shell. A complete solution of this problem, by means of asymptotic formulas derived from the shell theory, has been given by the writer in an earlier publication ("Spherical Shells Subjected to Axial Symmetrical Bending," *International Association for Bridge and Structural Engineering, Publications*, 5 [Zürich, 1938], 173–185).

There is still another approximate solution for the problem, introduced by J. W. Geckeler ("Über die Festigkeit achsen-symmetrischer Schalen," *Forschungsarbeiten auf dem Gebiete des Ingenieurwesens* [Heft 276], who has shown that for extremely thin shells with large angle of opening it is permissible to neglect all but the highest derivatives in the differential equation of bending. This approximation is equivalent to replacing the spherical shell with a cylindrical one of the same radius and wall thickness. Geckeler's formulas are somewhat less accurate than those derived above from the curved-beam theory and can be obtained from (149 a–g) by simply putting $\sin(\varphi_0 - \omega) = 1$. The expressions for the displacement and rotation of the edge, equations (150 a–c), are, however, the same in both cases.

Substituting this expression for Q in (a–g), we have the formulas for all the other unknown quantities in the shell problem:

$$\left.\begin{aligned}
M_1 &= \frac{r}{2\lambda} C \frac{e^{-\lambda\omega}}{\sqrt{\sin(\varphi_0 - \omega)}} [\cos(\lambda\omega + \psi) + \sin(\lambda\omega + \psi)], \\
M_2 &= \mu M_1 , \\
N &= C \frac{\lambda e^{-\lambda\omega}}{\sqrt{\sin(\varphi_0 - \omega)}} [\cos(\lambda\omega + \psi) - \sin(\lambda\omega + \psi)], \\
T &= - Q \cot(\varphi_0 - \omega), \\
u &= \frac{r \sin(\varphi_0 - \omega)}{Et} N, \\
\theta &= \frac{2\lambda^2}{Et} C \frac{e^{-\lambda\omega}}{\sqrt{\sin(\varphi_0 - \omega)}} \cos(\lambda\omega + \psi).
\end{aligned}\right\} \qquad (149\text{ b–g})$$

From the equations given the following formulas can be derived for the horizontal displacement u_0 and the angular deflection θ_0 of the edge of the shell ($\omega = 0$) due to unit values of edge loadings P_0 and M_0, respectively:

$$\left.\begin{aligned}
u_0^P &= \frac{2\lambda r \sin^2 \varphi_0}{Et}, \\
\theta_0^P = u_0^M &= \frac{2\lambda^2 \sin \varphi_0}{Et}, \\
\theta_0^M &= \frac{4\lambda^3}{Ert}.
\end{aligned}\right\} \qquad (150\text{ a–c})$$

Thus far we have discussed the problem of bending produced by forces and moments uniformly distributed along the base circle of the shell. This bending analysis forms, however, only part of the complete solution of the general problem of axially symmetrical deformation, when the shell may be loaded with distributed surface forces and subjected at the same time to certain boundary conditions required by the character of support at the edge of the shell. In such a general case the stress calculation will have to be resolved into two parts, the first of which is the membrane analysis and the second the bending analysis, in the manner described in previous chapters for cylindrical and conical shells.

In the first part of the analysis the shell is assumed to be a thin membrane, without any flexural resistance, exerting only normal forces, which are in equilibrium with the external loading distributed over the surface of the shell. Because of this membrane stress system, however, displacements and rotation take place at the edge of the shell which, in general, may not be compatible with the edge conditions required by the nature of the supports. In order to satisfy the boundary requirements forces and moments will have to be applied, as shown in the second part of the analysis, to the edge of the shell, thus producing bending which can then be calculated by the formulas developed above. So we see that for a complete analysis we still need to have the solution for the membrane problem, which will be developed briefly in the following paragraphs.

Let us assume that the surface of the shell is loaded with normal and meridional forces, Z and X, per unit area of the surface, which, being distributed symmetrically with respect to the axis of the shell, will be functions of the angle φ only. The membrane forces produced by this loading in the directions of the hoop and meridional circles will be denoted by N_m and T_m respectively and will be assumed to act per unit arc length of the corresponding circles. The positive directions for these components are shown in Figure 131.

Considering the equilibrium of the shell in the vertical direction, parallel to the axis, we find that along a hoop circle of the circumference $2\pi r \sin \varphi$ the vertical components of the meridional forces $T_m \sin \varphi$ will have to balance the sum of the vertical components of all surface forces between the hoop circle and the vertex of the shell. This requires that

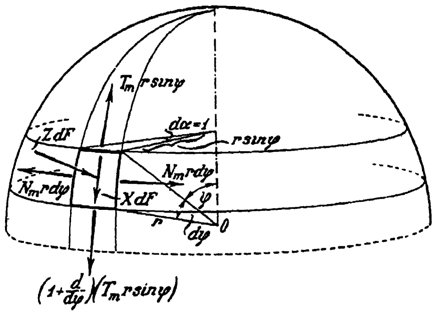

FIG. 131

$$2\pi r \sin \varphi T_m \sin \varphi = -\int_0^{\varphi} (X \sin \varphi + Z \cos \varphi) 2\pi r \sin \varphi \, r d\varphi,$$

which gives

$$T_m = -\frac{r}{\sin^2 \varphi} \int_0^{\varphi} (X \sin \varphi + Z \cos \varphi) \sin \varphi \, d\varphi. \tag{151a}$$

Once an expression is obtained for T_m, the other membrane force can be derived from the condition of equilibrium in a direction perpendicular to the surface of the shell, which can be stated as

$$\frac{1}{r}(N_m + T_m) = -Z. \tag{151b}$$

The displacements and rotations produced by this system of membrane forces can be calculated by the standard methods of the elastic theory.

Let us consider, for example, a spherical shell of radius $r = 90$ in., wall thickness $t = 3$ in., and angle of opening $2\varphi_0 = 70°$ subjected to a uniformly distributed radial pressure $p = 1$ lb./in.^{2} while the edge of the shell is rigidly fixed against any displacement or rotation (Fig. 132). For the material of the shell we shall assume reinforced concrete with a Poisson's ratio $\mu = \frac{1}{6}$.

If the shell is first regarded as a thin membrane free of any edge restraints,

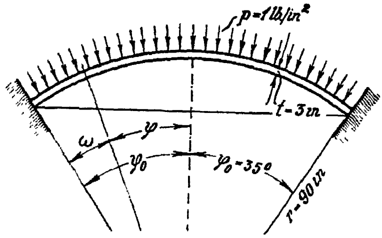

FIG. 132

its edge will, because of the constant membrane forces $N_m = T_m = pr/2 = -45$ lbs./in., move inward horizontally, without rotation, by the amount

$$u_0 = \frac{r \sin \varphi_0}{Et}(N_m - \mu T_m) = -\frac{645.0}{E} \text{ in.}$$

Since the rigid support of the shell does not permit any displacement of the edge, we shall have to superpose on the membrane stress system such edge loadings as will produce an outwardly directed horizontal displacement of the value u_0, without causing, however, rotation at the edge of the shell.

The condition that $[\theta]_{\omega=0} = 0$ defines the value of one of the integration constants, namely, the value of ψ in (149g), where we shall have $\psi = \pi/2$. The other integration constant, C, can be determined from the condition that in (149f)

$$[u]_{\omega=0} = -\frac{r \sin \varphi_0}{Et} C \frac{\lambda}{\sqrt{\sin \varphi_0}} = \frac{645.0}{E},$$

which gives, after putting $\lambda = \sqrt[4]{3(1 - \mu^2)r^2/t^2} = 7.16$,

$$C = -\frac{645.0t}{r\lambda\sqrt{\sin \varphi_0}} = -3.97 \text{ lbs./in.}$$

By substituting these values for C and ψ in (149 a–g) the forces, moments and displacements set up by the bending of the shell can readily be calculated. For the meridional bending moment M_1 and the hoop normal force N we thus have the following values:

$\omega°$	0	5	10	15	20	25	30	35
M_1 in. lbs./in.	−32.92	−4.28	6.96	8.20	5.64	2.68	0.50	∞
N lbs./in.	37.49	30.00	15.84	4.89	−0.92	−2.94	−3.15	∞

The final stress system in the shell will be obtained by a superposition of the bending and the membrane analysis. As is shown in the table above, the equations in (149 a–g) give infinite values for the unknown quantities at the vertex of the shell. This singularity is due to the asymptotic character of the approximation used, has influence only in the immediate neighborhood of the vertex, and can be disregarded in practical stress analysis.

For the sake of comparison the values of M_1 and N were calculated also by

Meissner's exact solution* for this problem, by means of hypergeometric series. The results of this calculation are as follows:

$\omega°$	0	5	10	15	20	25	30	35
M_1 in. lbs./in.	−37.68	−5.76	6.69	8.14	5.45	2.36	0.38	0.29
N lbs./in.	38.92	31.90	17.26	5.95	−0.02	−2.17	−2.50	−2.46

The distribution of the M_1 and N valuesalong the meridional circle, as obtained from the exact and the approximate solu tions, is shown in Figure 133.

The amount of numerical calculation needed for the exact solution was about twenty times more than that for the approximate solution. It is to be noted that the spherical shell in our example, with a ratio of $r/t = 30$, exemplifies one of the thickest types used in modern reinforced concrete structures. For thinner shells or for shells with a larger angle of opening the accuracy of the approximate solution is increased.

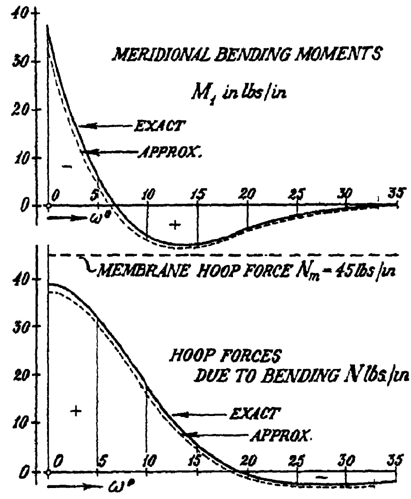

FIG. 133

50. *Approximate Solution for Flat Arches*

Frequently we find problems with the arch and the foundation of such proportions that in the expression

$$\eta = \sqrt[4]{\frac{r^4k}{EI} + 1}$$

the first term on the right side is large compared to unity. In such cases it is permissible to put in (141)

$$\alpha = \beta = \sqrt[4]{\frac{r^4k}{4EI}} = \lambda r = \rho, \qquad \text{(a)}$$

where λ denotes the same quantity used previously in solutions for bending of straight beams.

Dealing with flat arches, where the angle of opening $2\gamma < 60°$, we find that a further approximation is possible, namely, that we may also omit the term

* *Op. cit.* (see p. 167); see also Timoshenko, *op. cit.* (see p. 105).

y/r^2 in comparison with d^2y/dx^2 in the expression for M (equation [d] on p. 157). Thus we shall have

$$\left.\begin{aligned} M &= -\frac{EI}{r^2}\frac{d^2y}{d\varphi^2}, \\ Q &= -\frac{EI}{r^3}\frac{d^3y}{d\varphi^3}, \\ N &= rky + \frac{EI}{r^3}\frac{d^4y}{d\varphi^4}. \end{aligned}\right\} \tag{b}$$

On the basis of these simplifying assumptions we obtain from (142 a–d) and (143) the following general solutions for symmetrically loaded ($C_2 = C_3 = 0$) flat arches:

$$\left.\begin{aligned} y &= C_0 + C_1 \operatorname{Cosh} \rho\varphi \cos \rho\varphi + C_4 \operatorname{Sinh} \rho\varphi \sin \rho\varphi, \\ M &= 2\lambda^2 EI(C_1 \operatorname{Sinh} \rho\varphi \sin \rho\varphi - C_4 \operatorname{Cosh} \rho\varphi \cos \rho\varphi), \\ Q &= 2\lambda^3 EI[C_1(\operatorname{Cosh} \rho\varphi \sin \rho\varphi + \operatorname{Sinh} \rho\varphi \cos \rho\varphi) \\ &\qquad - C_4(\operatorname{Sinh} \rho\varphi \cos \rho\varphi - \operatorname{Cosh} \rho\varphi \sin \rho\varphi)], \\ N &= rkC_0, \\ \theta &= C_1\lambda(\operatorname{Sinh} \rho\varphi \cos \rho\varphi - \operatorname{Cosh} \rho\varphi \sin \rho\varphi) \\ &\qquad + C_4\lambda(\operatorname{Cosh} \rho\varphi \sin \rho\varphi + \operatorname{Sinh} \rho\varphi \cos \rho\varphi). \end{aligned}\right\} \tag{152 a–e}$$

For the horizontal displacement u we shall now have, in place of (144), the following formula:

$$u = -\rho[(C_1 - C_4)I_1 - (C_1 + C_4)I_2], \tag{152f}$$

where

$$I_1 = \frac{1}{4\rho}\left(\operatorname{Sinh} \rho\varphi\left[\frac{\rho}{\rho - 1}\cos(\rho - 1)\varphi - \frac{\rho}{\rho + 1}\cos(\rho + 1)\varphi\right] + \operatorname{Cosh} \rho\varphi\left[\sin(\rho - 1)\varphi - \sin(\rho + 1)\varphi\right]\right),$$

$$I_2 = \frac{1}{4\rho}\left(\operatorname{Cosh} \rho\varphi\left[\frac{\rho}{\rho + 1}\sin(\rho + 1)\varphi - \frac{\rho}{\rho - 1}\sin(\rho - 1)\varphi\right] - \operatorname{Sinh} \rho\varphi\left[\cos(\rho + 1)\varphi - \cos(\rho - 1)\varphi\right]\right).$$

A few applications of this approximate method will be shown below, by deriving formulas for flat arches subjected to symmetrical end loadings.

a. Equal Vertical Forces at the Two Ends (Fig. 134)

The end conditions for $\varphi = \gamma$ are

$$Q\cos\gamma + N\sin\gamma = P,$$

$$Q\sin\gamma - N\cos\gamma = 0,$$

and

$$M = 0.$$

Substituting here the general expression for M, Q, and N from (152 b–d), we have for the integration constants

$$C_0 = P\frac{\sin\gamma}{rk},$$

$$C_1 = P\frac{4\rho\cos\gamma}{rk}\,\frac{\operatorname{Cosh}\rho\gamma\cos\rho\gamma}{\operatorname{Sinh}2\rho\gamma + \sin 2\rho\gamma},$$

$$C_4 = P\frac{4\rho\cos\gamma}{rk}\,\frac{\operatorname{Sinh}\rho\gamma\sin\rho\gamma}{\operatorname{Sinh}2\rho\gamma + \sin 2\rho\gamma}.$$

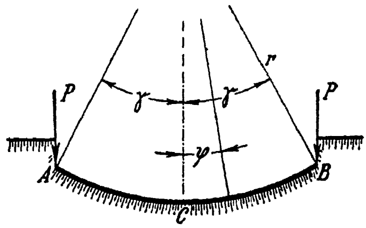

FIG. 134

Hence the equation for the radial deflection of the arch will be

$$y = \frac{P}{rk}\left\{\sin\gamma + \frac{2\rho\cos\gamma}{\operatorname{Sinh}2\rho\gamma + \sin 2\rho\gamma}\left[\operatorname{Cosh}\rho(\gamma+\varphi)\cos\rho(\gamma-\varphi) + \operatorname{Cosh}\rho(\gamma-\varphi)\cos\rho(\gamma+\varphi)\right]\right\}, \tag{153}$$

the radial deflection at the ends will be

$$y_A = y_B = \frac{P}{rk}\left(\sin\gamma + 2\rho\cos\gamma\,\frac{\operatorname{Cosh}2\rho\gamma + \cos 2\rho\gamma}{\operatorname{Sinh}2\rho\gamma + \sin 2\rho\gamma}\right),$$

and the deflection at the middle,

$$y_C = \frac{P}{rk}\left(\sin\gamma + 4\rho\cos\gamma\,\frac{\operatorname{Cosh}\rho\gamma\cos\rho\gamma}{\operatorname{Sinh}2\rho\gamma + \sin 2\rho\gamma}\right).$$

The deflection at the middle will vanish if

$$\tan\gamma = -4\rho\,\frac{\operatorname{Cosh}\rho\gamma\cos\rho\gamma}{\operatorname{Sinh}2\rho\gamma + \sin 2\rho\gamma}.$$

That value of $\rho\gamma$ which satisfies the equation above will determine the so-called effective length of the bar. It is seen that we must have a value of $2\rho\gamma > \pi$, which means that the effective length for curved beams will be larger than that for straight beams under the same loading conditions (see pp. 50, 54). If the radius r and the arc length $s = 2r\gamma$ of the arch and the modulus of the foundation k_0 ($k = bk_0$, from equation [a] on p. 2) are given, the condition $2\rho\gamma = \pi$ can be fulfilled by choosing the height of the arch h in such a way that

$$h = \frac{s}{\pi}\sqrt[3]{\frac{2k_0 s}{\pi E}} = 0.31s\sqrt[3]{\frac{k_0 s}{E}}. \qquad (154)$$

This value for h is proposed in the design of foundation arches.* With it, the deflection at the middle of the arch will be

$$y_C = P\frac{\sin\gamma}{rk};$$

while the deflection at the ends will be

$$y_A = y_B = \frac{P}{rk}\left(\sin\gamma + \frac{\pi}{\gamma}\cos\gamma \operatorname{Tanh}\frac{\pi}{2}\right).$$

At the same time the bending moment at the middle of the arch will have the following value:

$$M_C = -\frac{Ps\cos\gamma}{\pi \operatorname{Cosh}\frac{\pi}{2}} = -\frac{Ps\cos\gamma}{7.88}.$$

In the general case, for any value of h the bending moment at the middle of the arch can be obtained as

$$M_C = -P\frac{2r\cos\gamma}{\rho}\frac{\operatorname{Sinh}\rho\gamma\sin\rho\gamma}{\operatorname{Sinh}2\rho\gamma + \sin 2\rho\gamma}. \qquad (155)$$

b. Equal Horizontal Forces at the Two Ends (Fig. 135)

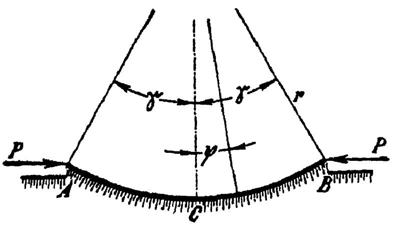

FIG. 135

The end conditions for $\varphi = \gamma$ are

$$N\sin\gamma + Q\cos\gamma = 0,$$

$$N\cos\gamma - Q\sin\gamma = P,$$

and

$$M = 0.$$

The conditions are satisfied by the following values of the integration constants:

$$C_0 = P\frac{\cos\gamma}{rk},$$

$$C_1 = -P\frac{4\rho\sin\gamma}{rk}\frac{\operatorname{Cosh}\rho\gamma\cos\rho\gamma}{\operatorname{Sinh}2\rho\gamma + \sin 2\rho\gamma},$$

$$C_4 = -P\frac{4\rho\sin\gamma}{rk}\frac{\operatorname{Sinh}\rho\gamma\sin\rho\gamma}{\operatorname{Sinh}2\rho\gamma + \sin 2\rho\gamma}.$$

* See Adolf Francke, "Einiges über Grundbögen," *Schweizerische Bauzeitung*, 36 (1900), 71.

Thus we have for the radial deflection of the arch the equation

$$y = \frac{P}{rk}\left\{\cos\gamma - \frac{2\rho\sin\gamma}{\operatorname{Sinh} 2\rho\gamma + \sin 2\rho\gamma}\left[\operatorname{Cosh}\rho(\gamma+\varphi)\cos\rho(\gamma-\varphi) + \operatorname{Cosh}\rho(\gamma-\varphi)\cos\rho(\gamma+\varphi)\right]\right\}. \quad (156)$$

Hence the radial deflection at the ends will be

$$y_A = y_B = \frac{P}{rk}\left(\cos\gamma - 2\rho\sin\gamma\,\frac{\operatorname{Cosh} 2\rho\gamma + \cos 2\rho\gamma}{\operatorname{Sinh} 2\rho\gamma + \sin 2\rho\gamma}\right),$$

and the deflection at the middle,

$$y_C = \frac{P}{rk}\left(\cos\gamma - 4\rho\sin\gamma\,\frac{\operatorname{Cosh}\rho\gamma\cos\rho\gamma}{\operatorname{Sinh} 2\rho\gamma + \sin 2\rho\gamma}\right).$$

At the same time the bending moment at the middle of the arch will be

$$M_C = P\,\frac{2r\sin\gamma}{\rho}\,\frac{\operatorname{Sinh}\rho\gamma\sin\rho\gamma}{\operatorname{Sinh} 2\rho\gamma + \sin 2\rho\gamma}. \quad (157)$$

c. Equal Concentrated Moments at the Two Ends (Fig. 136)

The end conditions at $\varphi = \gamma$, namely, $N = 0$, $Q = 0$, and $M = M_0$, give in this case the following values for the integration constants:

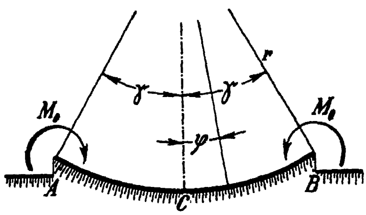

FIG. 136

$$C_0 = 0,$$

$$C_1 = -M_0\,\frac{4\rho^2}{r^2k}\,\frac{\operatorname{Sinh}\rho\gamma\cos\rho\gamma - \operatorname{Cosh}\rho\gamma\sin\rho\gamma}{\operatorname{Sinh} 2\rho\gamma + \sin 2\rho\gamma},$$

$$C_4 = -M_0\,\frac{4\rho^2}{r^2k}\,\frac{\operatorname{Sinh}\rho\gamma\cos\rho\gamma + \operatorname{Cosh}\rho\gamma\sin\rho\gamma}{\operatorname{Sinh} 2\rho\gamma + \sin 2\rho\gamma}.$$

Hence the equation for the deflection of the arch in the radial direction will be

$$y = -M_0\,\frac{4\rho^2}{r^2k}\,\frac{1}{\operatorname{Sinh} 2\rho\gamma + \sin 2\rho\gamma}\cdot\left[(\operatorname{Sinh}\rho\gamma\cos\rho\gamma - \operatorname{Cosh}\rho\gamma\sin\rho\gamma)\operatorname{Cosh}\rho\varphi\cos\rho\varphi + (\operatorname{Sinh}\rho\gamma\cos\rho\gamma + \operatorname{Cosh}\rho\gamma\sin\rho\gamma)\operatorname{Sinh}\rho\varphi\sin\rho\varphi\right]. \quad (158)$$

The radial deflection at the ends will be

$$y_A = y_B = -\frac{2M_0\rho^2}{r^2k}\,\frac{\operatorname{Sinh} 2\rho\gamma - \sin 2\rho\gamma}{\operatorname{Sinh} 2\rho\gamma + \sin 2\rho\gamma},$$

and the deflection at the middle,

$$y_C = -\frac{4M_0\rho^2}{r^2 k}\frac{\text{Sinh } \rho\gamma \cos \rho\gamma - \text{Cosh } \rho\gamma \sin \rho\gamma}{\text{Sinh } 2\rho\gamma + \sin 2\rho\gamma}.$$

The bending moment at the middle of the arch is obtained as

$$M_C = 2M_0\frac{\text{Sinh } \rho\gamma \cos \rho\gamma + \text{Cosh } \rho\gamma \sin \rho\gamma}{\text{Sinh } 2\rho\gamma + \sin 2\rho\gamma}. \tag{159}$$

51. *Corrugated Tubes*

It has been shown in § 11 that a longitudinal element of a cylindrical tube under axially symmetrical loading can always be regarded as a straight beam on an elastic foundation by taking the modulus of foundation $k = Et/R^2$, t being the thickness, R the radius of the tube, and E the modulus of elasticity of its material. On the basis of this analogy one may anticipate that circumferentially corrugated tubes, in which a longitudinal element is composed of a series of arches, can be analyzed by the flexural theory for curved beams on an elastic foundation. In the curved-beam theory developed in this chapter the foundation was assumed, however, to resist radial displacements of the circular arches, while in corrugated tubes the resistance of the foundation will be proportional to the displacement normal to the axis of the tube. The difference between these two types of foundation will be small only in flat arches, that is, for shallow corrugations of the tube, and therefore the investigation will be limited to such cases in the following discussion. In order to obtain sufficiently accurate results by this approximation it will be required that the angle of opening for each corrugation be such that $2\gamma < 60°$, and, furthermore, that the thickness of the tube t be small with respect to the radius of the corrugation r and that this quantity again be small compared to the radius of the tube R.

One of the main reasons for the technical application of corrugated tubes is that their flexibility and compressibility in the axial direction are many times larger than those of plain tubes. It will be shown below how this increased flexibility of corrugated tubes can be calculated by means of the general solution for elastically supported flat arches in (152 a–f).

Let us assume that the corrugation of the tube consists of a series of alternating concave and convex circular arches, as shown in Figure 137. Subjecting this tube to an axial compression P lbs./in. per unit length of the circumference of the tube, we find that because of the symmetry of the corrugation the end points of the arches, A and B, will not deflect normally to the axis of the tube; furthermore, that the bending moment will vanish at these end points and that the axial component of the normal force in the arches will balance the external loading P. The problem is thus reduced to the analysis of a circular arch (Fig. 138) supported on an elastic foundation, the modulus of which is $k = Et/R^2$, while at the ends of the arch the following conditions prevail:

$$[v]_{\varphi=\gamma} = 0 \quad \text{or} \quad [y - u \sin \varphi]_{\varphi=\gamma} = 0,$$

$$[M]_{\varphi=\gamma} = 0 \quad \text{and} \quad [N \cos \varphi - Q \sin \varphi]_{\varphi=\gamma} = P.$$

Substituting here the general expressions from (152 a–f), we can write the end conditions in the following form:

$$\left.\begin{aligned} C_0 + C_1 \operatorname{Cosh} \rho\gamma \cos \rho\gamma + C_4 \operatorname{Sinh} \rho\gamma \sin \rho\gamma & \\ + \rho \sin \gamma[(C_1 - C_4)I_1 - (C_1 + C_4)I_2] &= 0, \\ C_1 \operatorname{Sinh} \rho\gamma \sin \rho\gamma - C_4 \operatorname{Cosh} \rho\gamma \cos \rho\gamma &= 0, \\ rkC_0 \cos \gamma - \frac{k}{2\lambda} \sin \gamma[C_1(\operatorname{Cosh} \rho\gamma \sin \rho\gamma + \operatorname{Sinh} \rho\gamma \cos \rho\gamma) & \\ - C_4(\operatorname{Sinh} \rho\gamma \cos \rho\gamma - \operatorname{Cosh} \rho\gamma \sin \rho\gamma)] &= P. \end{aligned}\right\} \quad (160)$$

A simultaneous solution of these three equations will determine the integration constants C_0, C_1, and C_4. Substituting, then, these C values in the general

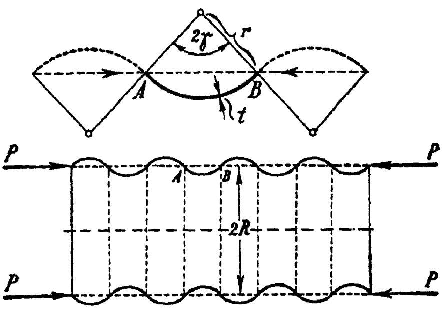

FIG. 137

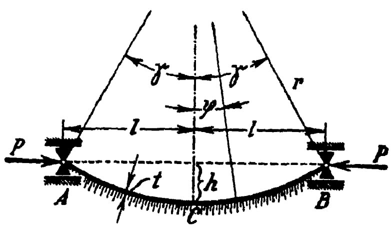

FIG. 138

solution, equations (152 a–f), we have the final formulas, from which all force and displacement components can be readily calculated.

The application of the method will be illustrated now by a numerical example requiring calculation of the flexibility of a corrugated tube of the following dimensions: $t = 0.065$ in., $R = 5.330$ in., $l = 0.670$ in., and $h = 0.235$ in. A tube with these dimensions was tested for flexibility by L. H. Donnell* and is used here in order to provide a direct comparison between experimental and theoretical results.

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* "The Flexibility of Corrugated Pipes under Longitudinal Forces and Bending," *Transactions of the American Society of Mechanical Engineers* Vol. 54 (1932), "Applied Mechanics Section," pp. 69–75. In this paper Donnell developed formulas for calculating the flexibility of triangular and semicircular corrugations by means of the minimum strain energy method, expressing the deflection line of the corrugations in the form of trigonometric series.

In the calculation we need the following constants, which are determined by the dimensions given for the tube: $r = 1.072$ in.; $\rho = \sqrt[4]{r^4k/4EI} = \sqrt[4]{3(1-\mu^2)r^4/R^2t^2} = 2.34$; $\lambda = \rho/r = 2.18$ in.$^{-1}$; $\gamma = 38°40' = 0.675$ radians, $\rho\gamma = 1.580$. With these data we have from (152f) $I_1 = 0.3810$ and $I_2 = 0.0865$, and hence can write up the end conditions in (160) in a numerical form, as follows:

$$C_0 + 0.408C_1 + 1.641C_4 = 0,$$

$$2.324C_1 + 0.023C_4 = 0,$$

$$C_0 - 0.428C_1 - 0.436C_4 = 522.0\,\frac{P}{E}.$$

The solution of these equations gives

$$C_0 = 413.0\,\frac{P}{E}, \quad C_1 = 2.5\,\frac{P}{E}, \quad \text{and} \quad C_4 = -252.2\,\frac{P}{E}.$$

Substituting these values in (152f), we have for the axial displacement component $[u]_{\varphi=\gamma} = -277.5\,P/E$, which means that the total span of one corrugation, $2l$, will shorten, because of the axial compression on the tube, by the amount $555.0\,P/E$. Since in a similar tube without corrugation the shortening of a $2l$ distance would be $(2l/t)(P/E)$, or $20.6\,P/E$, we find that the corrugations produced a 555.0/20.6, or 26.95-fold, increase in the flexibility of the tube. This increased flexibility can also be interpreted as if the modulus of elasticity of the steel tube were reduced from a value of $E = 30 \times 10^6$ lbs./in.^{2} for a smooth tube to a value of $E' = E/26.95 = 1.12 \times 10^6$ lbs./in.^{2} for the corrugated one. For this "reduced modulus of elasticity" of the tube Donnell* obtained experimentally $E' = 1.51 \times 10^6$ lbs./in.^{2} in axial compression, and in bending of the tube he found about the same value, getting $E' = 1.45 \times 10^6$ lbs./in.^{2} These values of E' are seen to be in fair agreement with the one obtained above from the theoretical solution.

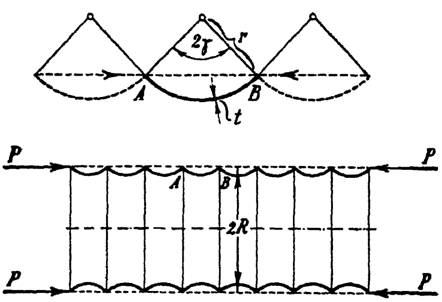

FIG. 139

The same method can be used for calculating the flexibility of other types of corrugations composed of flat circular arches; only the end conditions of the arches will then be of different character. Taking for instance the corrugation shown in Figure 139, we find that at the end of each arch the angular deflection and the component of the normal forces perpendicular to the axis of the tube will vanish, while the component of the normal force parallel to the axis of the tube will equal the external loading P.

* As cited on p. 177.

CHAPTER X

CONTINUITY IN THE FOUNDATION

52. *Partial Continuity: Foundation Layers*

In treating the various problems related to beams on elastic foundation we have thus far always assumed that the pressure in the foundation is proportional at every point to the deflection of the beam at that point and independent of the pressure or deflection occurring in other parts of the foundation. This assumption, which was first introduced by E. Winkler,* is mathematically by far the simplest that one can make regarding the nature of a supporting elastic medium. It assumes a complete lack of continuity in the material of the foundation, as if it consisted of a series of independent springs which deflect when directly loaded, but with no movement of the adjacent material (Fig. 140a). We have seen that there are cases, like that of the cylindrical tube subjected to axially symmetrical loading, when this simple assumption provides at the same time the mathematically rigorous formulation of the problem; in many other technical problems the solution developed on this basis, though not exact, is still the only theoretical approach for a stress analysis. This theory has proved adequate for calculating stresses and deflections in railroad tracks, but no particular claim has been made that the deformation or pressure distribution in actual earth foundations could be predicted by this method. The mechanical behavior of subsoils appears to be much more complex than that of any elastic material, and as yet it has been found impossible to establish any mathematical law which would conform with the observations made on these materials.† In some instances, as in the experiments of A. Föppl,‡ we may find subsoils of such character that the deformation in the foundation is localized mainly in the loaded region. In such cases, naturally, we may expect good agreement with a calculation based on Winkler's assumption. For the other limiting case, where the material of the foundation is completely continuous, we also have certain methods of solution available, which will be discussed in the latter part of this chapter.

But there is still a need to bridge the gap between these two extreme cases and to find a means of calculating foundations where the deformation is partly localized and partly continuous In the present section a method will be presented for treating problems of this nature. The partial continuity of the

* *Die Lehre von der Elastizität und Festigkeit* (Prag, 1867), p. 182.

† See O. K. Fröhlich, *Druckverteilung im Baugrunde* (Vienna, 1934).

‡ *Vorlesungen über technische Mechanik* (9th ed.; Leipzig, 1922), III, 258.

foundation will be realized by assuming a continuous beam imbedded in the material of the foundation, which is itself without any continuity. When such a foundation is loaded by a distributed load q over a short section dx we find that the deflection under the load will consist of a discontinuous part y_1 and also of a continuous deflection curve y_2, as shown in Figure 140b. It is seen that by proper selection of the foundation moduli k_1, k_2 and the bending rigidity of the beam within the foundation EI_2 any ratio between y_1 and y_2 can be reproduced by such a mechanical model, while the total compressibility of the foundation $1/k_1 + 1/k_2$ may still be kept constant. The bending analysis of beams supported on such a composite foundation will be discussed below.

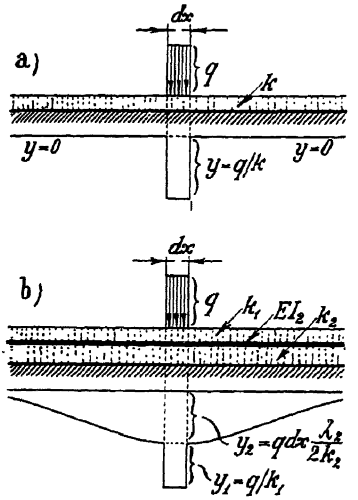

FIG. 140

Let us assume that the upper and the lower layers of the foundation have the respective moduli k_1 and k_2, that the beam imbedded between these two layers has a bending rigidity EI_2, and that this entire foundation is resting on a rigid base and supports a beam of bending rigidity EI_1 subjected to a number of concentrated loads (Fig. 141). Denoting the deflection ordinates of the upper and lower beams by y_1 and y_2 respectively, we find that the distributed pressure in the foundation under the upper beam is* $p_1 = k_1(y_1 - y_2)$ and, under the lower beam, $p_2 = k_2 y_2$, while the resultant pressure acting on the lower beam is $p_2 - p_1 = (k_1 + k_2)y_2 - k_1 y_1$. Since, according to our assumption, these pressures form the only distributed loading on the two beams, we have from the flexural theory

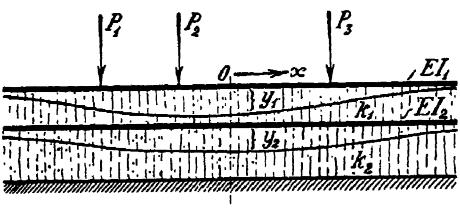

FIG. 141

$$EI_1 \frac{d^4 y_1}{dx^4} = -p_1 = -k_1(y_1 - y_2), \tag{a}$$

$$EI_2 \frac{d^4 y_2}{dx^4} = -(p_2 - p_1) = -(k_1 + k_2)y_2 + k_1 y_1. \tag{b}$$

* The dimensions of k_1 and k_2 are in lbs. per in.^{2} and those of p_1 and p_2 in lbs. per in.

The pressures in these cases are considered positive when accompanied by positive (downward) deflections. From (a) we have

$$y_2 = \frac{EI_1}{k_1}\frac{d^4y_1}{dx^4} + y_1 \tag{c}$$

or

$$\frac{d^4y_2}{dx^4} = \frac{EI_1}{k_1}\frac{d^8y_1}{dx^8} + \frac{d^4y_1}{dx^4}. \tag{d}$$

If we substitute these expressions in (b), y_2 will cancel out, and we shall have the differential equation of the elastic line of the upper beam:

$$\frac{d^8y_1}{dx^8} + A\frac{d^4y_1}{dx^4} + By_1 = 0, \tag{161}$$

where

$$A = \frac{k_1(I_1 + I_2) + k_2 I_1}{EI_1 I_2} \quad \text{and} \quad B = \frac{k_1}{EI_1}\frac{k_2}{EI_2}.$$

The general solution of the equation above can be obtained by putting $y_1 = e^{mx}$. The result of this substitution is a characteristic equation of the eighth order, the roots of which are

$$m_{1,2,3,4} = (\pm 1 \pm i)\sqrt[4]{\frac{\alpha + \beta}{4}} \quad \text{and} \quad m_{5,6,7,8} = (\pm 1 \pm i)\sqrt[4]{\frac{\alpha - \beta}{4}},$$

where

$$\alpha = \frac{A}{2}, \qquad \beta = \sqrt{\frac{A^2}{4} - B}.$$

Thus, by introducing the notation

$$\sqrt[4]{\frac{\alpha + \beta}{4}} = \lambda_1 \quad \text{and} \quad \sqrt[4]{\frac{\alpha - \beta}{4}} = \lambda_2,$$

the general solution of (161) can be written in the following form:

$$\begin{aligned} y_1 = {} & e^{\lambda_1 x}(C_1 \cos\lambda_1 x + C_2 \sin\lambda_1 x) + e^{-\lambda_1 x}(C_3 \cos\lambda_1 x + C_4 \sin\lambda_1 x) \\ & + e^{\lambda_2 x}(C_5 \cos\lambda_2 x + C_6 \sin\lambda_2 x) + e^{-\lambda_2 x}(C_7 \cos\lambda_2 x + C_8 \sin\lambda_2 x). \end{aligned} \tag{162}$$

Substituting this expression in (c), we have the general solution for the deflection line of the beam within the foundation:

$$\begin{aligned} y_2 = {} & \left[1 - (\alpha + \beta)\frac{EI_1}{k_1}\right][e^{\lambda_1 x}(C_1 \cos\lambda_1 x + C_2 \sin\lambda_1 x) \\ & \qquad + e^{-\lambda_1 x}(C_3 \cos\lambda_1 x + C_4 \sin\lambda_1 x)] \\ & + \left[1 - (\alpha - \beta)\frac{EI_1}{k_1}\right][e^{\lambda_2 x}(C_5 \cos\lambda_2 x + C_6 \sin\lambda_2 x) \\ & \qquad + e^{-\lambda_2 x}(C_7 \cos\lambda_2 x + C_8 \sin\lambda_2 x)]. \end{aligned} \tag{163}$$

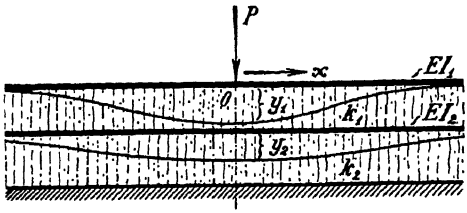

FIG. 142

The general solution obtained above will now be applied to a particular case, when both beams are of unlimited length and the upper one is subjected to a concentrated force P at $x = 0$ (Fig. 142). Here the condition that at $x = \infty$ the deflections and all their derivatives in both beams must vanish requires that we put in (162) and (163)

$$C_1 = C_2 = C_5 = C_6 = 0.$$

The four remaining integration constants are determined by the conditions at $x = 0$, where we have

$$1.\quad \left(\frac{dy_1}{dx}\right) = 0; \qquad 2.\quad \left(\frac{dy_2}{dx}\right) = 0;$$

$$3.\quad \left(EI_1\frac{d^3y_1}{dx^3}\right) = \frac{P}{2}; \qquad 4.\quad \left(EI_2\frac{d^3y_2}{dx^3}\right) = 0.$$

The first two of these conditions, when expressed in terms of the derivatives of (162), require that

$$\lambda_1(-C_3 + C_4) + \lambda_2(-C_7 + C_8) = 0$$

and

$$\left[1 - (\alpha + \beta)\frac{EI_1}{k_1}\right]\lambda_1(-C_3 + C_4) + \left[1 - (\alpha - \beta)\frac{EI_1}{k_1}\right]\lambda_2(-C_7 + C_8) = 0,$$

which give

$$C_3 = C_4 \quad \text{and} \quad C_7 = C_8 .$$

Hence the number of integration constants is reduced to two, which are then defined by conditions 3 and 4 above. We thus get

$$C_3 = \frac{P}{16\,EI_1\beta\lambda_1^3}\left[\frac{k_1}{EI_1} - (\alpha - \beta)\right]$$

and

$$C_7 = -\frac{P}{16\,EI_1\beta\lambda_2^3}\left[\frac{k_1}{EI_1} - (\alpha + \beta)\right].$$

Substituting these C values in (162) and (163), we have the final expressions for the elastic line of both beams. The successive derivatives of these deflection lines will define also the angular deflection θ, the bending moment M, and the shearing force Q along the beams. Thus in Figure 142 we have the complete solution for the problem in the following form:

Upper beam:

$$\left.\begin{aligned}
y_1 &= \frac{P}{16EI_1\beta}\left[D_1\frac{e^{-\lambda_1 x}}{\lambda_1^3}(\cos\lambda_1 x + \sin\lambda_1 x)\right. \\
&\qquad \left. - D_2\frac{e^{-\lambda_2 x}}{\lambda_2^3}(\cos\lambda_2 x + \sin\lambda_2 x)\right], \\
\theta_1 &= \frac{P}{8EI_1\beta}\left(-D_1\frac{e^{-\lambda_1 x}}{\lambda_1^2}\sin\lambda_1 x + D_2\frac{e^{-\lambda_2 x}}{\lambda_2^2}\sin\lambda_2 x\right), \\
M_1 &= \frac{P}{8\beta}\left[D_1\frac{e^{-\lambda_1 x}}{\lambda_1}(\cos\lambda_1 x - \sin\lambda_1 x)\right. \\
&\qquad \left. - D_2\frac{e^{-\lambda_2 x}}{\lambda_2}(\cos\lambda_2 x - \sin\lambda_2 x)\right], \\
Q_1 &= \frac{P}{4\beta}\left(-D_1 e^{-\lambda_1 x}\cos\lambda_1 x + D_2 e^{-\lambda_2 x}\cos\lambda_2 x\right).
\end{aligned}\right\} \quad (164\text{ a–d})$$

Lower beam:

$$\left.\begin{aligned}
y_2 &= -\frac{P}{16EI_2\beta}\frac{k_1}{EI_1}\left[\frac{e^{-\lambda_1 x}}{\lambda_1^3}(\cos\lambda_1 x + \sin\lambda_1 x)\right. \\
&\qquad \left. - \frac{e^{-\lambda_2 x}}{\lambda_2^3}(\cos\lambda_2 x + \sin\lambda_2 x)\right], \\
\theta_2 &= \frac{P}{8EI_2\beta}\frac{k_1}{EI_1}\left(\frac{e^{-\lambda_1 x}}{\lambda_1^2}\sin\lambda_1 x - \frac{e^{-\lambda_2 x}}{\lambda_2^2}\sin\lambda_2 x\right), \\
M_2 &= -\frac{P}{8\beta}\frac{k_1}{EI_1}\left[\frac{e^{-\lambda_1 x}}{\lambda_1}(\cos\lambda_1 x - \sin\lambda_1 x)\right. \\
&\qquad \left. - \frac{e^{-\lambda_2 x}}{\lambda_2}(\cos\lambda_2 x - \sin\lambda_2 x)\right], \\
Q_2 &= \frac{P}{4\beta}\frac{k_1}{EI_1}(e^{-\lambda_1 x}\cos\lambda_1 x - e^{-\lambda_2 x}\cos\lambda_2 x).
\end{aligned}\right\} \quad (165\text{ a–d})$$

The distribution of pressure under the upper and lower beams, p_1 and p_2 respectively, can be calculated as $p_1 = k_1(y_1 - y_2)$ and $p_2 = k_2 y_2$. The notations in the formulas above have the following values:

$$D_1 = \frac{k_1}{EI_1} - (\alpha - \beta), \qquad D_2 = \frac{k_1}{EI_1} - (\alpha + \beta);$$

$$\alpha = \frac{A}{2}, \qquad \beta = \sqrt{\frac{A^2}{4} - B};$$

$$A = \frac{k_1(I_1 + I_2) + k_2 I_1}{EI_1 I_2}, \qquad B = \frac{k_1}{EI_1}\frac{k_2}{EI_2};$$

and

$$\lambda_1 = \sqrt[4]{\frac{\alpha + \beta}{4}}, \qquad \lambda_2 = \sqrt[4]{\frac{\alpha - \beta}{4}}.$$

The formulas in (164 a–d) and (165 a–d) give the values of the unknown quantities along the beam to the right of the point of the application of the load ($x > 0$). The sign convention adopted here is the same as that used previously for the simple beam (see p. 11). The curves for y and M are even functions of x, while θ and Q are odd functions, just as they were for the simple beam in Figure 5. From the solution above for a single concentrated load it is possible to derive, by superposition, any combination of concentrated or distributed loadings on the infinitely long upper beam. Similarly, solution for beams of finite length can be obtained by determining the integration constants in (162) and (163) in such a manner as is required by the nature of the problem.

In order to show by an example the effect of the partial continuity discussed above, let us consider the case when the upper layer of the foundation is subjected to the action of an infinitely rigid punch loaded by a uniform pressure q. Assuming that the rigid punch has an unlimited extension in one direction, we can calculate the pressure distribution under the corner of the punch in a particularly simple manner. Let us first suppose that the beam within the foundation is cut through in the vertical plane at the corner of the punch (Fig. 143). The result of this will be that the part of the beam under the punch will displace vertically by an amount $\Delta y = q/k_2$, while the other part of the beam will remain at rest. Since the nature of the problem necessitates the continuity in the foundation beam, we shall have to apply at the point of separation concentrated forces P_0 and moments M_0 in order to reëstablish the continuity in deflection and slope at this point of the beam. The problem is thus reduced to the calculation of semi-infinite beams under end loadings, which can be carried out by means of (19 a–d) and (20 a–d). The part of the beam under the punch will have a modulus of foundation $k_a = k_1 + k_2$, while the other part will have the modulus $k_b = k_2$. Denoting $\sqrt[4]{(k_1 + k_2)/4EI}$ by λ_a, and $\sqrt[4]{k_2/4EI}$

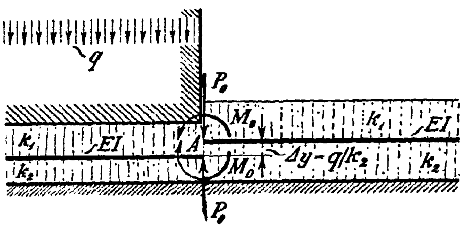

FIG. 143

by λ_b, we can write the conditions of continuity for displacement and slope at point A in the following form:

$$2P_0\left(\frac{\lambda_a}{k_a}+\frac{\lambda_b}{k_b}\right)-2M_0\left(\frac{\lambda_a^2}{k_a}-\frac{\lambda_b^2}{k_b}\right)=\frac{q}{k_b},$$

$$P_0\left(\frac{\lambda_a^2}{k_a}-\frac{\lambda_b^2}{k_b}\right)-2M_0\left(\frac{\lambda_a^3}{k_a}+\frac{\lambda_b^3}{k_b}\right)=0.$$

Determining the values of P_0 and M_0 from the equations above and calculating the change of pressure distribution in the foundation due to the action of P_0 and M_0, we find that the pressure under the corner of the rigid punch is considerably increased as a result of the continuity produced in the material of the foundation by the imbedded beam.

Taking for a numerical example $k_1 = 30$ lbs./in.^{2}, $k_2 = 1$ lb./in.^{2}, $EI = 100$ lbs. in.^{2}, and $q = 1$ lb./in., we have $P_0 = 2.670$ lbs. and $M_0 = -3.450$ in. lbs. and find that for such proportions in the foundation the pressure at the corner of the rigid punch will be 5.58 times larger than the average loading q (Fig. 144).

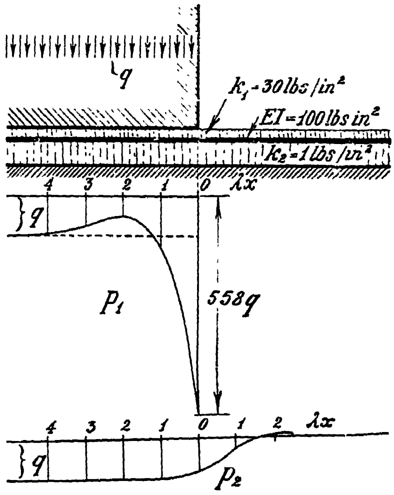

FIG. 144

53. *Interconnected Girders*

Let us consider a construction consisting of two parallel main girders supported on a series of cantilever cross beams, as shown in Figure 145, and assume that under loading the main girders will deflect only in the vertical principal plane of their cross section, thus resisting any twisting action which may be due to their rigid connection with the cross beams. Under such conditions each span (d_1 and d_2) of the crossbeams will act separately as a beam restrained against rotations at both ends, whereas the ends are displaced vertically with respect to each other by an amount corresponding to the difference in the deflection between the two main girders or the second girder and the built-in ends of the crossbeams respectively. The flexibility of each crossbeam can be characterized by the amount of load necessary to produce a unit relative deflection between the two ends of the beam. This load under the given conditions, for a beam of length d and flexural rigidity EI_0, will be equivalent to $12EI_0/d^3$. If the crossbeams are sufficiently closely spaced, their resistance can be replaced by distributed reaction forces acting along the main girders. The intensity of this distributed reaction p will be proportional at every point to the relative displacement of the ends of the crossbeams Δy at that point, $p = k\Delta y$, the proportionality factor per unit length of the main girder being $k = 12EI_0/cd^3$, where c is the spacing of the crossbeams.

Denoting by y_1 the deflection of the outer main girder of flexural rigidity EI_1 and by y_2 the deflection of the inner main girder of flexural rigidity EI_2, we find that the distributed elastic resistance of the crossbeams on each of these girders will be $p_1 = k_1(y_1 - y_2)$ and $p_2 = k_2y_2$, respectively, where $k_1 = 12EI_0/cd_1^3$ and $k_2 = 12EI_0/cd_2^3$. On the assumption that the external loading on the girders consists of concentrated forces, the only distributed loading will be the

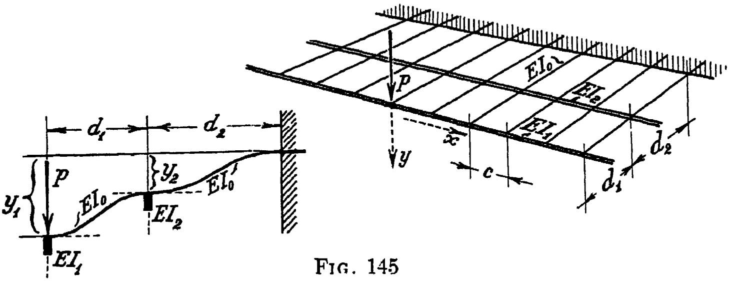

FIG. 145

pressures p_1 and p_2 derived above, and hence we can write, according to the flexural theory of beams,

$$EI_1 \frac{d^4 y_1}{dx^4} = -p_1 = -k_1(y_1 - y_2)$$

and

$$EI_2 \frac{d^4 y_2}{dx^4} = -(p_2 - p_1) = -(k_1 + k_2)y_2 + k_1 y_1.$$

It is seen that these equations are identical with equations (a) and (b) of the previous section (p. 180), from which it follows that (164 a–d) and (165 a–d) will be the solution also for the structural problem shown in Figure 145, in which the inner beam does not have any concentrated load.

The analogy between interconnected girders and layers of beams and foundations is a quite general one and can be extended to a variety of problems, including the system of beams in the hulls of ships* and the structure of longitudinal and cross girders applied in bridge constructions.† As an illustration of the application of the method we will discuss now the solution of a problem of the latter type.

Let us consider a bridge construction, consisting of four main girders and a large number of closely spaced crossbeams, loaded by a concentrated force P_0 on one of the inner girders, as shown in Figure 146. The ends of the crossbeams

* Application of this method to ship constructions was first made by J. G. Boobnov, *Theory of Structure of Ships* (St. Petersburg, 1913) (in Russian). See also W. Schilling, *Statik der Bodenkonstruktion der Schiffe* (Berlin, 1925).

† See A. J. S. Pippard and J. P. A. De Waele, "The Loading of Interconnected Bridge-Girders," *Journal of the Institution of Civil Engineers* (London, 1938), pp. 97–114. The example in § 53 is taken from this publication.

will be assumed to be rigidly connected with the main girders. Denoting the deflection of the four girders by y_1, y_2, y_3 and y_4 and assuming that the two outer girders and the two inner ones are in pairs of the same flexural rigidity, EI_1 and EI_2, respectively, we have, according to the theory developed above, the following expressions for the distributed reaction of the crossbeams along each of the main girders:

$$p_1 = EI_1 \frac{d^4 y_1}{dx^4} = k(y_2 - y_1),$$

$$p_2 = EI_2 \frac{d^4 y_2}{dx^4} = k(y_1 - 2y_2 + y_3),$$

$$p_3 = EI_2 \frac{d^4 y_3}{dx^4} = k(y_2 - 2y_3 + y_4),$$

$$p_4 = EI_1 \frac{d^4 y_4}{dx^4} = k(y_3 - y_4),$$

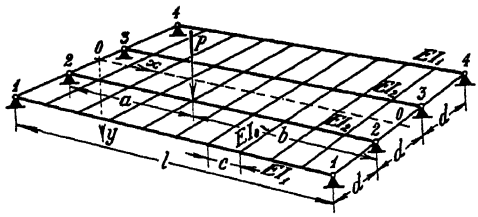

FIG. 146

where $k = 12EI_0/cd^3$, d being the span, c the spacing and EI_0 the flexural rigidity of the crossbeams.

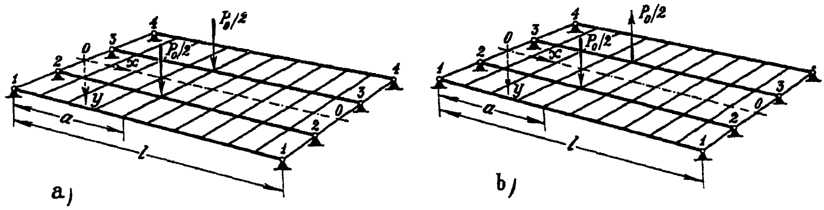

FIG. 147

The solution of these four simultaneous equations can be considerably simplified by resolving the load P_0 into symmetrical and antisymmetrical components in the manner shown in Figures 147 a and b, respectively.

It is seen that in the symmetrical loading we have

$$y_1 = y_4 \quad \text{and} \quad y_2 = y_3,$$

while in the antisymmetrical case

$$y_1 = -y_4 \quad \text{and} \quad y_2 = -y_3,$$

which means that the solution for each of the loading components will involve only two simultaneous differential equations. The solutions for these symmetrical and antisymmetrical loading components will be discussed separately below. By superposing these two cases the solution for the general case of Figure 146 may be obtained.

a. Symmetrical Case

From the equations

$$EI_1 \frac{d^4 y_1}{dx^4} = k(y_2 - y_1)$$

and

$$EI_2 \frac{d^4 y_2}{dx^4} = k(y_1 - y_2)$$

we have, after elimination of y_2,

$$\frac{d^8 y_1}{dx^8} + \frac{k}{E}\left(\frac{1}{I_1} + \frac{1}{I_2}\right)\frac{d^4 y_1}{dx^4} = 0. \tag{166}$$

By using the notation

$$\lambda = \sqrt[4]{\frac{k}{4E}\left(\frac{1}{I_1} + \frac{1}{I_1}\right)},$$

we can write the general solution of (166) in the following form:

$$y_1 = A + Bx + Cx^2 + Dx^3 + E \operatorname{Cosh} \lambda x \cos \lambda x + F \operatorname{Sinh} \lambda x \sin \lambda x + G \operatorname{Cosh} \lambda x \sin \lambda x + H \operatorname{Sinh} \lambda x \cos \lambda x, \tag{167}$$

while at the same time for the inner beams we have

$$y_2 = A + Bx + Cx^2 + Dx^3 - \frac{I_1}{I_2}(E \operatorname{Cosh} \lambda x \cos \lambda x + F \operatorname{Sinh} \lambda x \sin \lambda x + G \operatorname{Cosh} \lambda x \sin \lambda x + H \operatorname{Sinh} \lambda x \cos \lambda x). \tag{168}$$

The eight integration constants occurring in these equations can be determined from the conditions of the system. Since the deflection and bending moment above the hinged end supports of the main girders must vanish we have

$$A = C = E = F = 0.$$

The other four constants are determined by the requirements that deflections, slopes, and bending moments be continuous through the loaded section ($x = a$) of the girders, while at the same time in the shearing force of the inner beams there be a step of $P_0/2$ at the points of application of the loads. The integration constants satisfying these conditions have the following values:

$$B = -\frac{P_0 ab(l + b)}{12lE(I_1 + I_2)}, \qquad D = \frac{P_0 b}{12lE(I_1 + I_2)},$$

$$G = \frac{P_0}{2} \frac{\begin{matrix}\text{Cosh}\,\lambda l \sin \lambda l(\text{Sinh}\,\lambda b \cos \lambda b - \text{Cosh}\,\lambda b \sin \lambda b) \\ - \text{Sinh}\,\lambda l \cos \lambda l(\text{Sinh}\,\lambda b \cos \lambda b + \text{Cosh}\,\lambda b \sin \lambda b)\end{matrix}}{4\lambda^3 E(I_1 + I_2)(\text{Cosh}^2 \lambda l - \cos^2 \lambda l)},$$

$$H = \frac{P_0}{2} \frac{\begin{matrix}\text{Sinh}\,\lambda l \cos \lambda l(\text{Sinh}\,\lambda b \cos \lambda b - \text{Cosh}\,\lambda b \sin \lambda b) \\ + \text{Cosh}\,\lambda l \sin \lambda l(\text{Sinh}\,\lambda b \cos \lambda b + \text{Cosh}\,\lambda b \sin \lambda b)\end{matrix}}{4\lambda^3 E(I_1 + I_2)(\text{Cosh}^2 \lambda l - \cos^2 \lambda l)}.$$

With these values for the integration constants the expressions for the deflection y, bending moment M, shearing force Q, and distributed reaction p along the outer and the inner girders (1 and 2 respectively) will be, for values of $x < a$, as follows:

$$\left.\begin{aligned}
y_1 &= \frac{P_0 bx[x^2 - a(l + b)]}{12lE(I_1 + I_2)} + G\,\text{Cosh}\,\lambda x \sin \lambda x + H\,\text{Sinh}\,\lambda x \cos \lambda x, \\
M_1 &= \frac{P_0 bx}{2l}\left(\frac{I_1}{I_1 + I_2}\right) + 2\lambda^2 EI_1(G\,\text{Sinh}\,\lambda x \cos \lambda x \\
&\qquad - H\,\text{Cosh}\,\lambda x \sin \lambda x), \\
Q_1 &= \frac{P_0 b}{2l}\left(\frac{I_1}{I_1 + I_2}\right) + 2\lambda^3 EI_1[G(\text{Cosh}\,\lambda x \cos \lambda x - \text{Sinh}\,\lambda x \sin \lambda x) \\
&\qquad - H(\text{Cosh}\,\lambda x \cos \lambda x + \text{Sinh}\,\lambda x \sin \lambda x)], \\
p_1 &= -4\lambda^4 EI_1(G\,\text{Cosh}\,\lambda x \sin \lambda x + H\,\text{Sinh}\,\lambda x \cos \lambda x);
\end{aligned}\right\} \quad (169\text{ a–d})$$

$$\left.\begin{aligned}
y_2 &= \frac{P_0 bx[x^2 - a(l + b)]}{12lE[I_1 + I_2)} - \frac{I_1}{I_2}(G\,\text{Cosh}\,\lambda x \sin \lambda x \\
&\qquad + H\,\text{Sinh}\,\lambda x \cos \lambda x), \\
M_2 &= \frac{P_0 bx}{2l}\left(\frac{I_2}{I_1 + I_2}\right) - 2\lambda^2 EI_1(G\,\text{Sinh}\,\lambda x \cos \lambda x \\
&\qquad - H\,\text{Cosh}\,\lambda x \sin \lambda x), \\
Q_2 &= \frac{P_0 b}{2l}\left(\frac{I_2}{I_1 + I_2}\right) - 2\lambda^3 EI_1[G(\text{Cosh}\,\lambda x \cos \lambda x - \text{Sinh}\,\lambda x \sin \lambda x) \\
&\qquad - H(\text{Cosh}\,\lambda x \cos \lambda x + \text{Sinh}\,\lambda x \sin \lambda x)], \\
p_2 &= -p_1 = 4\lambda^4 EI_1(G\,\text{Cosh}\,\lambda x \sin \lambda x + H\,\text{Sinh}\,\lambda x \cos \lambda x).
\end{aligned}\right\} \quad (170\text{ a–d})$$

The solution above can be used also for values of $x > a$, if a and b are interchanged and x is replaced by $x' = l - x$.

b. Antisymmetrical Case

From the equations

$$EI_1 \frac{d^4 y_1}{dx^4} = k(y_2 - y_1)$$

and

$$EI_2 \frac{d^4 y_2}{dx^4} = k(y_1 - 3y_2),$$

after eliminating y_2, we get for y_1

$$\frac{d^8 y_1}{dx^8} + \frac{k}{E}\left(\frac{1}{I_1} + \frac{3}{I_2}\right)\frac{d^4 y_1}{dx^4} + \frac{2k^2}{E^2 I_1 I_2} y_1 = 0. \tag{171}$$

The general solution of this equation by means of the notations

$$\alpha = \frac{I_1}{I_2}, \qquad \beta = \sqrt[4]{(1 + 3\alpha + \sqrt{9\alpha^2 - 2\alpha + 1})\frac{k}{8EI_1}},$$

and

$$\gamma = \sqrt[4]{(1 + 3\alpha - \sqrt{9\alpha^2 - 2\alpha + 1})\frac{k}{8EI_1}}$$

can be written in the following form:

$$\begin{aligned} y_1 = {} & L \operatorname{Cosh} \beta x \cos \beta x + M \operatorname{Sinh} \beta x \sin \beta x + N \operatorname{Cosh} \beta x \sin \beta x \\ & + P \operatorname{Sinh} \beta x \cos \beta x + Q \operatorname{Cosh} \gamma x \cos \gamma x + R \operatorname{Sinh} \gamma x \sin \gamma x \\ & + S \operatorname{Cosh} \gamma x \sin \gamma x + T \operatorname{Sinh} \gamma x \cos \gamma x; \end{aligned} \tag{172}$$

while for y_2 we have from above

$$\begin{aligned} y_2 = {} & \left(1 - \frac{4\beta^4 EI_1}{k}\right)(L \operatorname{Cosh} \beta x \cos \beta x + M \operatorname{Sinh} \beta x \sin \beta x \\ & + N \operatorname{Cosh} \beta x \sin \beta x + P \operatorname{Sinh} \beta x \cos \beta x) \\ & + \left(1 - \frac{4\gamma^4 EI_1}{k}\right)(Q \operatorname{Cosh} \gamma x \cos \gamma x + R \operatorname{Sinh} \gamma x \sin \gamma x \\ & + S \operatorname{Cosh} \gamma x \sin \gamma x + T \operatorname{Sinh} \gamma x \cos \gamma x). \end{aligned} \tag{173}$$

The conditions for the determination of the integration constants will be the same now as in the symmetrical case, and thus we get

$$L = M = Q = R = 0,$$

$$\frac{N}{(G)_\beta} = \frac{P}{(H)_\beta} = \frac{k\lambda^3}{2\beta^3(\beta^4 - \gamma^4)} \frac{I_1 + I_2}{EI_1 I_2},$$

$$\frac{S}{(G)_\gamma} = \frac{T}{(H)_\gamma} = -\frac{k\lambda^3}{2\gamma^3(\beta^4 - \gamma^4)} \frac{I_1 + I_2}{EI_1 I_2},$$

where the symbols $(G)_\beta$, $(H)_\beta$, $(G)_\gamma$, and $(H)_\gamma$ denote the same expressions, G and H, as in the symmetrical case above, except that λa, λb, λl are now replaced by βa, βb, βl or γa, γb, γl, according to the subscript.

The final solution for the outer and inner main girders (1 and 2 respectively) of the antisymmetrically loaded structure in Figure 147b will thus be for values of $x < a$

$$\left.\begin{aligned}
y_1 &= N \operatorname{Cosh}\beta x \sin\beta x + P \operatorname{Sinh}\beta x \cos\beta x + S \operatorname{Cosh}\gamma x \sin\gamma x \\
&\quad + T \operatorname{Sinh}\gamma x \cos\gamma x, \\
M_1 &= 2EI_1[\beta^2(N \operatorname{Sinh}\beta x \cos\beta x - P \operatorname{Cosh}\beta x \sin\beta x) \\
&\quad + \gamma^2(S \operatorname{Sinh}\gamma x \cos\gamma x - T \operatorname{Cosh}\gamma x \sin\gamma x)], \\
Q_1 &= 2EI_1\{\beta^3[N(\operatorname{Cosh}\beta x \cos\beta x - \operatorname{Sinh}\beta x \sin\beta x) \\
&\quad - P(\operatorname{Cosh}\beta x \cos\beta x + \operatorname{Sinh}\beta x \sin\beta x)] \\
&\quad + \gamma^3[S(\operatorname{Cosh}\gamma x \cos\gamma x - \operatorname{Sinh}\gamma x \sin\gamma x) \\
&\quad - T(\operatorname{Cosh}\gamma x \cos\gamma x + \operatorname{Sinh}\gamma x \sin\gamma x)]\}, \\
p_1 &= -4EI_1[\beta^4(N \operatorname{Cosh}\beta x \sin\beta x + P \operatorname{Sinh}\beta x \cos\beta x) \\
&\quad + \gamma^4(S \operatorname{Cosh}\gamma x \sin\gamma x + T \operatorname{Sinh}\gamma x \cos\gamma x)];
\end{aligned}\right\} \quad (174\ \text{a–d})$$

$$\left.\begin{aligned}
y_2 &= \left(1 - \frac{4EI_1\beta^4}{k}\right)(N \operatorname{Cosh}\beta x \sin\beta x + P \operatorname{Sinh}\beta x \cos\beta x) \\
&\quad + \left(1 - \frac{4EI_1\gamma^4}{k}\right)(S \operatorname{Cosh}\gamma x \sin\gamma x + T \operatorname{Sinh}\gamma x \cos\gamma x), \\
M_2 &= 2EI_2\left[\beta^2\left(1 - \frac{4EI_1\beta^4}{k}\right)(N \operatorname{Sinh}\beta x \cos\beta x - P \operatorname{Cosh}\beta x \sin\beta x)\right. \\
&\quad \left. + \gamma^2\left(1 - \frac{4EI_1\gamma^4}{k}\right)(S \operatorname{Sinh}\gamma x \cos\gamma x - T \operatorname{Cosh}\gamma x \sin\gamma x)\right], \\
Q_2 &= 2EI_2\left[\beta^3\left(1 - \frac{4EI_1\beta^4}{k}\right)[N(\operatorname{Cosh}\beta x \cos\beta x - \operatorname{Sinh}\beta x \sin\beta x)\right. \\
&\quad - P(\operatorname{Cosh}\beta x \cos\beta x + \operatorname{Sinh}\beta x \sin\beta x)] \\
&\quad + \gamma^3\left(1 - \frac{4EI_1\gamma^4}{k}\right)[S(\operatorname{Cosh}\gamma x \cos\gamma x - \operatorname{Sinh}\gamma x \sin\gamma x) \\
&\quad \left. - T(\operatorname{Cosh}\gamma x \cos\gamma x + \operatorname{Sinh}\gamma x \sin\gamma x)]\right], \\
p_2 &= -4EI_2\left[\beta^4\left(1 - \frac{4EI_1\beta^4}{k}\right)(N \operatorname{Cosh}\beta x \sin\beta x\right. \\
&\quad + P \operatorname{Sinh}\beta x \cos\beta x) \\
&\quad \left. + \gamma^4\left(1 - \frac{4EI_1\gamma^4}{k}\right)(S \operatorname{Cosh}\gamma x \sin\gamma x + T \operatorname{Sinh}\gamma x \cos\gamma x)\right].
\end{aligned}\right\} \quad (175\ \text{a–d})$$

By superposition we can obtain from the symmetrical and antisymmetrical cases above the solution for the deflection line of the main girders 1, 2, 3, and 4 in the general case of Figure 146 as follows:

$$y_1 = (y_1)_{\text{sym.}} + (y_1)_{\text{antisym.}}$$

$$y_2 = (y_2)_{\text{sym.}} + (y_2)_{\text{antisym.}}$$

$$y_3 = (y_2)_{\text{sym.}} - (y_2)_{\text{antisym.}}$$

$$y_4 = (y_1)_{\text{sym.}} - (y_2)_{\text{antisym.}}$$

The superposition of bending moments, shearing forces, and crossbeam reactions can be carried out in the same manner.

For the case when the load P_0 is on the outer girder, girder number 1, at a distance a from the left support, the solution will be of the same form as above, equations (169 a–d) and (170 a–d) for the symmetrical and (174 a–d) and (175 a–d) for the antisymmetrical case, only the expressions for the integration constants will have to be modified in those equations. The relationship between these new constants (marked by a prime) and the ones obtained above will be as follows:

a. Symmetrical Case

$$B' = B, \qquad D' = D, \qquad G' = -G\frac{I_2}{I_1}, \qquad H' = -H\frac{I_2}{I_1}. \tag{176}$$

b. Antisymmetrical Case

$$\left.\begin{aligned} \frac{N'}{(G')_\beta} &= \frac{P'}{(H')_\beta} = -\frac{\lambda^3(I_1 + I_2)}{2\beta^3(\beta^4 - \gamma^4)I_1}\left(\frac{k}{EI_1} - 4\gamma^4\right), \\ \frac{S'}{(G')_\gamma} &= \frac{T'}{(H')_\gamma} = \frac{\lambda^3(I_1 + I_2)}{2\gamma^3(\beta^4 - \gamma^4)I_1}\left(\frac{k}{EI_1} - 4\beta^4\right). \end{aligned}\right\} \tag{177}$$

54. *Grillage Beams*

As a counterpart to the subject of the previous section, where we have discussed rigidly interconnected girders and crossbeams, let us investigate the problem of a grillage of beams, where the longitudinal girders are freely supported on crossbeams and no bending or twisting moment is transmitted at these points from one beam to the other. In the analysis of this type of problem it is convenient to express the deflection lines of the girders in the form of trigonometric series, on the basis of the minimum strain-energy principle, applications of which have already been shown in previous chapters.

The deflection line of a simply supported beam under any loading condition is known to be expressible in the form

$$y = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l}. \tag{a}$$

Assuming that this deflection line is produced by a distributed loading p (Fig. 148), which can also be represented as

$$p = \sum_{n=1}^{\infty} p_n \sin \frac{n\pi x}{l}, \qquad \text{(b)}$$

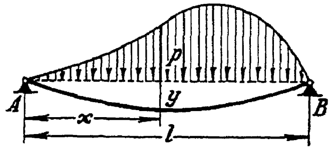

FIG. 148

we find, since $p = EI(d^4y/dx^4)$, that in the equations above

$$p_n = \frac{EI\pi^4}{l^4} n^4 a_n \qquad \text{and} \qquad a_n = \frac{l^4}{EI\pi^4} \frac{1}{n^4} p_n .$$

Substituting these expressions in (a) and (b) respectively, we have

$$\left. \begin{aligned} y &= \frac{l^4}{\pi^4 EI} \sum_{n=1}^{\infty} \frac{1}{n^4} p_n \sin \frac{n\pi x}{l}, \\ p &= \frac{\pi^4 EI}{l^4} \sum_{n=1}^{\infty} n^4 a_n \sin \frac{n\pi x}{l}. \end{aligned} \right\} \qquad \text{(178 a–b)}$$

These two equations represent a general interrelation between loading and deflection in any hinged-end beam and can be conveniently used for the analysis of grillage beams, as will be shown below.*

Let us consider first a simple grillage, consisting of two main girders freely supported on a larger number of crossbeams, the ends of which have hinged supports along the line O–O, as shown in Figure 149.

It will be assumed that the spacing of the crossbeams, c, is sufficiently close for the concentrated reactions of the crossbeams to be replaced by continuously distributed loadings along the main girders. The intensity of the reaction forces produced at the hinged ends of the crossbeams as the grillage is subjected to external loading will be denoted by p. This p as a function of x will be regarded as the statically unknown quantity in the present problem, and its determination will be carried out in the same manner as that generally used in solving statically indeterminate structures.

First we shall assume that the hinged supports are removed from the ends of the crossbeams, and thus the main girders can freely deflect without any restraint on the part of the crossbeams (statically determinate fundamental system). If the outer girder, number 1, in this system is loaded with a concentrated force P, the crossbeams will remain straight, and their farther end will describe in the O–O vertical plane a deflection e/d times the inverted deflection line of girder 1. We thus get $y_0^P = -(e/d)y_1^P$, where the upper index

* This approach has been used by the writer in a paper entitled "A Method of Calculating Grillage Beams," *Stephen Timoshenko 60th Anniversary Volume* (New York, 1938). A solution for the case when a grillage of beams is supported on all its four sides was developed by Timoshenko, "Über die Biegung von Trägerrosten," *Zeitschrift für angewandte Mathematik und Mechanik*, 13 (1933), 153–159.

denotes the cause of deflection and the lower index shows the place where the deflection is produced.

Using the known formula expressing in sine series the deflection of a simply supported beam under a load P at a distance a from the support, we have

$$y_0^P = -\frac{e}{d}\frac{2Pl^3}{\pi^4 EI_1}\sum_{n=1}^{\infty}\frac{1}{n^4}\sin\frac{n\pi a}{l}\sin\frac{n\pi x}{l}. \tag{c}$$

This would be the deflection line in the O–O plane if the ends of the crossbeams were free to move. But in the structure under consideration (Fig. 149) the ends of the crossbeams are hinged; therefore deflection at these points will not occur, but distributed reaction forces p will be originated instead. These p forces are to cancel the y_0^P deflection curve along the O–O line, and their value can be determined from this condition, that $y_0^P = y_0^p$, where y_0^p denotes the deflection of the free ends of the crossbeams in the fundamental system under the action of the unknown p forces.

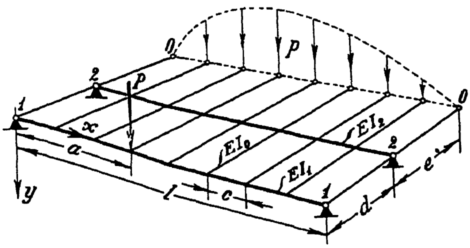

FIG. 149

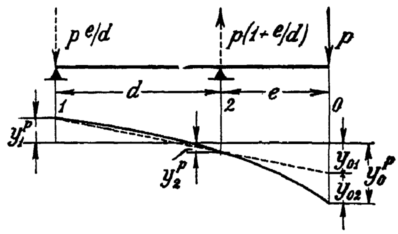

FIG. 150

The y_0^p deflection consists of two parts, one caused by the deflection of the main girders 1 and 2, the other due to the elastic deformation of the crossbeams loaded by p, as is shown in Figure 150:

$$y_0^p = y_{01} + y_{02},$$

where

$$y_{01} = y_1^p\frac{e}{d} + y_2^p\frac{d+e}{d}$$

and

$$y_{02} = p\frac{c}{3EI_0}e^2(d+e).$$

Putting $p = \sum p_n \sin n\pi x/l$, we can, by means of (178a), write the total y_0^p deflection in the form

$$y_0^p = \sum_{n=1}^{\infty}\left\{\frac{1}{n^4}\frac{l^4}{\pi^4 E}\left[\frac{e^2}{d^2}\frac{1}{I_1} + \left(1+\frac{e}{d}\right)^2\frac{1}{I_2}\right] + \frac{c}{3EI_0}e^2(d+e)\right\}p_n\sin\frac{n\pi x}{l}. \tag{d}$$

Equating from (c) and (d) $y_0^P = y_0^p$, we have the solution for the unknown reaction p as

$$p = -\sum_{n=1}^{\infty} \frac{P \dfrac{2de}{l} \dfrac{1}{I_1} \sin \dfrac{n\pi a}{l} \sin \dfrac{n\pi x}{l}}{e^2 \dfrac{1}{I_1} + (d+e)^2 \dfrac{1}{I_2} + n^4 \dfrac{\pi^4}{l^4} \dfrac{c}{3I_0} d^2 e^2 (d+e)}. \tag{179}$$

It is seen that the negative sign in the formula above denotes an upward-acting reaction p. As these p forces along the O–O line are known, the problem becomes a statically determinate one. The components of p on the main girders 1 and 2 will be $p_1 = p(e/d)$ and $p_2 = -p(1 + e/d)$ respectively. Shearing force, bending moment, slope and deflection curves due to the loading p can be obtained by successive integration of (179). By integration the convergence of the series increases, and in most cases the first few terms will give an accuracy sufficient in technical applications.

The solution for the reaction forces p can also be obtained, in this same manner, when the concentrated load P is on girder 2 at a distance a from the left support. Here we have

$$p = \sum_{n=1}^{\infty} \frac{1}{D} P \frac{2d(d+e)}{l} \frac{1}{I_2} \sin \frac{n\pi a}{l} \sin \frac{n\pi x}{l}, \tag{180}$$

where D stands for the same denominator as that in (179). The loading p will now point downward (positive direction) along the O–O line of supports. The components of p on girders 1 and 2 are $p_1 = p(e/d)$ and $p_2 = -p \cdot (1 + e/d)$.

Thus far we have investigated the cases when it was required that the deflection of the crossbeams along the O–O line be zero. In a similar manner let us now consider the problem when the ends of the crossbeams are subjected to such restraints along the O–O line as will keep the slope at the ends horizontal ($\theta = 0$). Such an end condition can be satisfied by applying along the O–O line at the ends of the crossbeams distributed moments $m = \sum m_n \sin n\pi x/l$, which will cancel the slopes originating at those ends in the fundamental system, where the crossbeams are free to displace under the action of the concentrated load P (Fig. 151). The condition from which the unknown moments m can be determined will now be

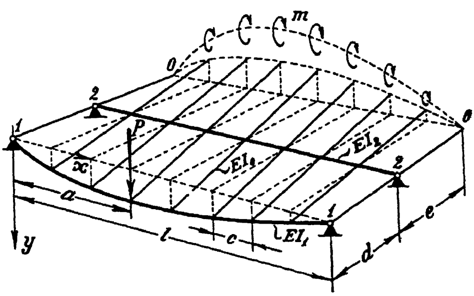

FIG. 151

$$\theta_0^P = \theta_0^m,$$

where θ_0^P and θ_0^m denote the angular deflection of the ends of the crossbeams in the statically fundamental system under the action of P and m respectively.

Assuming that P acts on girder 1, we have

$$\theta_0^P = -\frac{1}{d}y_1^P = -\frac{1}{d}\frac{2Pl^3}{\pi^4 EI_1}\sum_{n=1}^{\infty}\frac{1}{n^4}\sin\frac{n\pi a}{l}\sin\frac{n\pi x}{l}. \tag{e}$$

The expression for θ_0^m will contain terms representing the effect of the relative deflection between the girders 1 and 2 and the flexibility of the crossbeams. Using the notations in Figure 152, we shall have

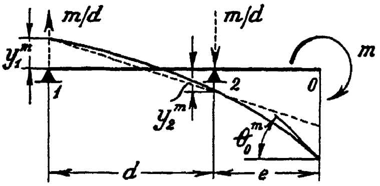

FIG. 152

$$\theta_0^m = \frac{1}{d}(y_1^m + y_2^m) + m\frac{c(d+3e)}{3EI_0}.$$

Putting now $m = \sum m_n \sin n\pi x/l$ and expressing y_1^m and y_2^m by means of (178a), we have

$$\theta_0^m = \sum_{n=1}^{\infty}\left[\frac{1}{d^2}\frac{l^4}{\pi^4 E}\left(\frac{1}{I_1}+\frac{1}{I_2}\right)\frac{1}{n^4} + \frac{c(d+3e)}{3EI_0}\right] m_n \sin\frac{n\pi x}{l}. \tag{f}$$

Equating from (e) and (f) $\theta_0^P = \theta_0^m$, we have for the unknown m the solution

$$m = -\sum_{n=1}^{\infty}\frac{P\dfrac{2d}{l}\dfrac{1}{I_1}\sin\dfrac{n\pi a}{l}\sin\dfrac{n\pi x}{l}}{\dfrac{1}{I_1}+\dfrac{1}{I_2}+n^4\dfrac{\pi^4}{l^4}\dfrac{c}{3I_0}d^2(d+3e)}. \tag{181}$$

In this formula, m is considered positive if causing tension in the lower fibers of the crossbeams. The distributed reactions due to these m moments on girders 1 and 2 will be $p_1 = (1/d)m$ and $p_2 = -(1/d)m$, respectively, the downward-directed force always being taken as positive.

For the case when the force P is on girder 2 at a distance a from the left end the restraining moments m, distributed along the O–O line are obtained as

$$m = \sum_{n=1}^{\infty}\frac{1}{D}P\frac{2d}{l}\frac{1}{I_2}\sin\frac{n\pi a}{l}\sin\frac{n\pi x}{l}, \tag{182}$$

where D denotes the same denominator as that in (181).

The two problems investigated above and shown in Figures 149 and 151 can be regarded as parts of the solution for the general case, where a four-beam symmetrical grillage is loaded by an arbitrarily placed concentrated force P (Fig. 153). Resolving the load P into two components, a symmetrical and an antisymmetrical one with respect to the axis of symmetry O–O of the grillage, we find that for the antisymmetrical case the

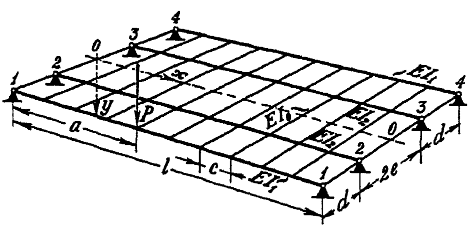
FIG. 153

deflection and the bending moment in the crossbeams will be zero along O–O, whereas in the symmetrical case the slope and the shearing force will vanish at these points. These two cases are evidently identical with the two whose solutions we obtained in (179–180) and (181–182). By superposition, therefore, we can now derive the distributed reaction forces of the crossbeams p_1, p_2, p_3, and p_4 along each of the main girders 1, 2, 3, and 4 respectively for any arbitrarily placed transverse loading on the symmetrical four-girder grillage. Thus we get for the case when the load P is on girder 1 at a distance a from the left support

$$\left.\begin{aligned}
p_1 &= \frac{P}{l}\sum_{n=1}^{\infty}\frac{1}{I_1}\left(-\frac{e^2}{D_1}-\frac{1}{D_2}\right)\sin\frac{n\pi a}{l}\sin\frac{n\pi x}{l},\\
p_2 &= \frac{P}{l}\sum_{n=1}^{\infty}\frac{1}{I_1}\left[\frac{e(d+e)}{D_1}+\frac{1}{D_2}\right]\sin\frac{n\pi a}{l}\sin\frac{n\pi x}{l},\\
p_3 &= \frac{P}{l}\sum_{n=1}^{\infty}\frac{1}{I_1}\left[-\frac{e(d+e)}{D_1}+\frac{1}{D_2}\right]\sin\frac{n\pi a}{l}\sin\frac{n\pi x}{l},\\
p_4 &= \frac{P}{l}\sum_{n=1}^{\infty}\frac{1}{I_1}\left(\frac{e^2}{D_1}-\frac{1}{D_2}\right)\sin\frac{n\pi a}{l}\sin\frac{n\pi x}{l},
\end{aligned}\right\} \qquad (183\text{a–d})$$

where

$$D_1 = e^2\frac{1}{I_1} + (d+e)^2\frac{1}{I_2} + n^4\frac{\pi^4}{l^4}\frac{c}{3I_0}d^2e^2(d+e)$$

and

$$D_2 = \frac{1}{I_1} + \frac{1}{I_2} + n^4\frac{\pi^4}{l^4}\frac{c}{3I_0}d^2(d+3e).$$

With the same notations as above we have for the case when the load P is on girder 2 the following expressions:

$$\left.\begin{aligned}
p_1 &= \frac{P}{l}\sum_{n=1}^{\infty}\frac{1}{I_2}\left[\frac{e(d+e)}{D_1}+\frac{1}{D_2}\right]\sin\frac{n\pi a}{l}\sin\frac{n\pi x}{l},\\
p_2 &= \frac{P}{l}\sum_{n=1}^{\infty}\frac{1}{I_2}\left[-\frac{(d+e)^2}{D_1}-\frac{1}{D_2}\right]\sin\frac{n\pi a}{l}\sin\frac{n\pi x}{l},\\
p_3 &= \frac{P}{l}\sum_{n=1}^{\infty}\frac{1}{I_2}\left[\frac{(d+e)^2}{D_1}-\frac{1}{D_2}\right]\sin\frac{n\pi a}{l}\sin\frac{n\pi x}{l},\\
p_4 &= \frac{P}{l}\sum_{n=1}^{\infty}\frac{1}{I_2}\left[-\frac{e(d+e)}{D_1}+\frac{1}{D_2}\right]\sin\frac{n\pi a}{l}\sin\frac{n\pi x}{l}.
\end{aligned}\right\} \qquad (184\text{a–d})$$

55. *Complete Continuity: Elastic Solid Foundation*

The nature of a supporting elastic medium of any type can best be described by the deflection line of its surface under a unit concentrated load. In the major part of the present work we have assumed the Winkler type of foundation, in which a unit load causes a completely discontinuous deflection line (Fig. 154a). In the last three sections a modification of this theory was discussed in which the deflection line of the foundation under a unit load consisted of a continuous and a discontinuous part (Fig. 154b). We have now to consider the case in which there is a complete continuity in the material of the foundation, the deflection line (Fig. 154c) being a continuous function K of the absolute difference of the coordinates of the point of application of the load ξ and the place x, where the deflection is produced:

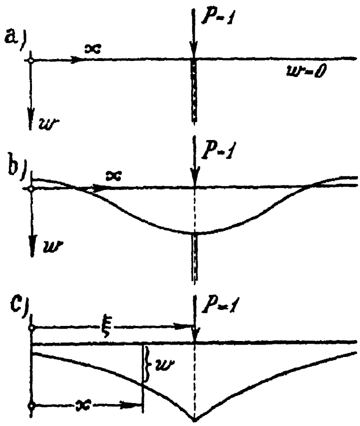

FIG. 154

$$w = K(|x - \xi|).$$

If a foundation, characterized by this function, is subjected to any distributed pressure $p(\xi)$ along the length l of a beam, the deflection line of the surface of the foundation is obtained as

$$w = \int_0^l p(\xi)K(|x - \xi|)\, d\xi.$$

Assuming that the deflection of the beam under consideration is the same along the length l as that of the foundation, if the loading on the beam is denoted by $q(x)$ we have from the flexural theory of beams

$$\frac{d^4 w}{dx^4} = C[q(x) - p(x)].$$

By combining the last two equations we may have the problem stated in the form of an integrodifferential equation. This approach has been employed by K. Wieghardt,* the first to investigate the problem of a continuous elastic foundation, and he has shown that, in the special case when

$$K(|x - \xi|) = e^{-c(x-\xi)},$$

the method yields solution for beams of finite length under certain definite conditions of loading. It is to be noted, however, that the type of foundation

* "Über den Balken auf nachgiebiger Unterlage," *Zeitschrift für angewandte Mathematik und Mechanik*, 2 (1922), 165–184. This method of solution has also been discussed in the following publications: P. Neményi, "Tragwerke auf elastischer Unterlage," *Zeitschrift für angewandte Mathematik und Mechanik*, 11 (1931), 450–463; E. Reissner, "On the Theory of Beams Resting on a Yielding Foundation," *Proceedings of the National Academy of Sciences*, 23 (1937), 328–333.

corresponding to Wieghardt's assumption is not identical with a homogeneous and isotropic elastic solid, since in the latter we have*

$$K(|x-\xi|) = C_1 \log |x-\xi| + C_2,$$

and with this expression for K the solution of the integrodifferential equation mentioned above has not as yet been obtained.

Problems of the bending of beams supported on an elastic solid were first solved by W. Prager,† and his method of solution, based on applying functions of a complex variable, will be discussed in more detail below.

Assume that an elastic beam of height h and of unlimited length is supported on an elastic solid foundation and loaded along its upper edge by a distributed periodic load

$$q(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}. \tag{185}$$

The coordinate axes for the beam and the foundation will be denoted by x, y and x_1, y_1 respectively, as shown in Figure 155, and it will be assumed that the extension of the beam, the foundation, and also of the load, is unlimited in the z direction, so that the problem is that of plane strain. In the derivation we shall consider a unit width of the beam and foundation in the z direction. Denoting by σ_x and σ_y the normal stresses, by u and v the displacement components in the x and y directions respectively, and by τ_{xy} the shearing stress in this coordinate system, we shall assume that

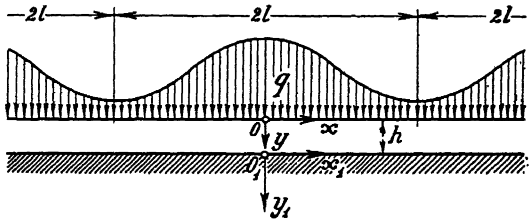

FIG. 155

at $y = 0$, $\tau_{xy} = 0$ and $\sigma_y = -q$,

at $y = h$, and $y_1 = 0$, $\tau_{xy} = \tau_{x_1y_1} = 0$,

$\sigma_y = \sigma_{y_1}$, and $v = v_1$,

while for

$$y_1 = \infty$$

stresses and deformations must remain finite.

The equilibrium equations in a two-dimensional stress system in the directions x and y require that

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \quad \text{and} \quad \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = 0. \tag{a}$$

* See S. Timoshenko, *Theory of Elasticity* (New York, 1934), p. 87.

† "Zur Theorie elastisch gelagerter Konstruktionen," *Zeitschrift für angewandte Mathematik und Mechanik,* 7 (1927), 354–360.

These equations are satisfied if the stress components are expressed as derivatives of a stress function F in the following manner:

$$\sigma_x = \frac{\partial^2 F}{\partial y^2}, \qquad \sigma_y = \frac{\partial^2 F}{\partial x^2}, \qquad \tau_{xy} = -\frac{\partial^2 F}{\partial x\, \partial y}. \tag{b}$$

Denoting by e the volume change and by ω the rotation, we have

$$e = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y},$$

$$\omega = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right),$$

and

$$\sigma_x + \sigma_y = \Delta F = \frac{E}{(1 + \mu)(1 - 2\mu)}\, e, \tag{c}$$

where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ and E and μ are the modulus of elasticity and Poisson's ratio respectively. The equilibrium equations above can also be expressed, by e and ω, in terms of derivatives of the displacements u and v; we thus get:

$$\frac{1 - \mu}{1 - 2\mu}\frac{\partial e}{\partial x} - \frac{\partial \omega}{\partial y} = 0,$$

$$\frac{1 - \mu}{1 - 2\mu}\frac{\partial e}{\partial y} + \frac{\partial \omega}{\partial x} = 0.$$

Hence we see that e and ω are conjugate harmonic functions; consequently, they can be regarded as real and imaginary parts, U and V, of a function of a complex variable $z = x + iy$:

$$\frac{1 - \mu}{1 - 2\mu}\, e + i\omega = U + iV. \tag{d}$$

Assuming that

$$\int (U + iV)\, dz = \varphi + i\psi, \tag{e}$$

we have

$$\frac{\partial\varphi}{\partial x} = \frac{\partial\psi}{\partial y} = U = (1 - \mu)\Delta F,$$

$$\frac{\partial\varphi}{\partial y} = -\frac{\partial\psi}{\partial x} = -V,$$

and thus

$$\left.\begin{aligned} u &= \frac{\partial F}{\partial x} + \varphi - s \\ v &= -\frac{\partial F}{\partial y} + \psi - t, \end{aligned}\right\} \tag{f}$$

where s and t are also conjugate harmonic functions.

Applying the differential operator Δ to (c), we find that the stress function F is biharmonic:

$$\Delta\Delta F = 0. \tag{g}$$

Having obtained the general equations of elasticity above, let us return now to (185), where it was assumed that the loading q on the beam is periodical, with a period of $2l$. In such a case we find that, along the verticals, $x = 0$, $x = \pm 2l$, $x = \pm 4l$, etc., the horizontal displacement u and its first derivative $\partial u/\partial x$ must vanish. A stress function F which satisfies these requirements can be assumed in the following form:

$$F(x, y) = \sum_{n=1}^{\infty} f_n(y) \cos\frac{n\pi x}{l}. \tag{186}$$

Substituting this expression for F in (g), we have an ordinary differential equation for $f_n(y)$:

$$\frac{d^4 f_n(y)}{dy^4} - 2\frac{n^2\pi^2}{l^2}\frac{d^2 f_n(y)}{dy^2} + \frac{n^4\pi^4}{l^4} f_n(y) = 0,$$

the solution of which is

$$f_n(y) = A_n \operatorname{Cosh}\frac{n\pi y}{l} + B_n \operatorname{Sinh}\frac{n\pi y}{l} + C_n y \operatorname{Cosh}\frac{n\pi y}{l} + D_n y \operatorname{Sinh}\frac{n\pi y}{l}. \tag{187}$$

Since the function F in this form can satisfy the boundary requirements regarding stresses and deformations, these conditions will have to be satisfied independently also by the harmonic functions s and t in (f). Thus we have for the stress and displacement components in the beam

$$
\left.
\begin{aligned}
\sigma_x &= \sum_{n=1}^{\infty} \gamma_n^2 \cos \gamma_n x \Big[A_n \operatorname{Cosh} \gamma_n y + B_n \operatorname{Sinh} \gamma_n y \\
&\qquad + C_n \left(y \operatorname{Cosh} \gamma_n y + \frac{2}{\gamma_n} \operatorname{Sinh} \gamma_n y \right) \\
&\qquad\qquad + D_n \left(y \operatorname{Sinh} \gamma_n y + \frac{2}{\gamma_n} \operatorname{Cosh} \gamma_n y \right) \Big], \\
\sigma_y &= -\sum_{n=1}^{\infty} \gamma_n^2 \cos \gamma_n x (A_n \operatorname{Cosh} \gamma_n y + B_n \operatorname{Sinh} \gamma_n y \\
&\qquad + C_n y \operatorname{Cosh} \gamma_n y + D_n y \operatorname{Sinh} \gamma_n y), \\
\tau_{xy} &= \sum_{n=1}^{\infty} \gamma_n^2 \sin \gamma_n x \Big[A_n \operatorname{Sinh} \gamma_n y + B_n \operatorname{Cosh} \gamma_n y \\
&\qquad + C_n \left(y \operatorname{Sinh} \gamma_n y + \frac{1}{\gamma_n} \operatorname{Cosh} \gamma_n y \right) \\
&\qquad\qquad + D_n \left(y \operatorname{Cosh} \gamma_n y + \frac{1}{\gamma_n} \operatorname{Sinh} \gamma_n y \right) \Big], \\
u &= \frac{1+\mu}{E} \sum_{n=1}^{\infty} \gamma_n \sin \gamma_n x \Big[A_n \operatorname{Cosh} \gamma_n y + B_n \operatorname{Sinh} \gamma_n y \\
&\qquad + C_n \left(y \operatorname{Cosh} \gamma_n y + \frac{2(1-\mu)}{\gamma_n} \operatorname{Sinh} \gamma_n y \right) \\
&\qquad + D_n \left(y \operatorname{Sinh} \gamma_n y + \frac{2(1-\mu)}{\gamma_n} \operatorname{Cosh} \gamma_n y \right) \Big], \\
v &= -\frac{1+\mu}{E} \sum_{n=1}^{\infty} \gamma_n \cos \gamma_n x \Big[A_n \operatorname{Sinh} \gamma_n y + B_n \operatorname{Cosh} \gamma_n y \\
&\qquad + C_n \left(y \operatorname{Sinh} \gamma_n y - \frac{1-2\mu}{\gamma_n} \operatorname{Cosh} \gamma_n y \right) \\
&\qquad + D_n \left(y \operatorname{Cosh} \gamma_n y - \frac{1-2\mu}{\gamma_n} \operatorname{Sinh} \gamma_n y \right) \Big],
\end{aligned}
\right\} \quad (188\text{ a–e})
$$

here $\gamma_n = n\pi/l$.

In these equations the integration constants satisfying the boundary conditions assumed in the problem will be

$$A_n = \frac{a_n}{\gamma_n^2},$$

$$B_n = -\frac{a_n}{\gamma_n^2} \frac{k \operatorname{Sinh}^2 \gamma_n h + \operatorname{Sinh} \gamma_n h \operatorname{Cosh} \gamma_n h + \gamma_n h}{k(\operatorname{Sinh} \gamma_n h \operatorname{Cosh} \gamma_n h + \gamma_n h) + \operatorname{Sinh}^2 \gamma_n h - \gamma_n^2 h^2},$$

$$C_n = -\gamma_n B_n,$$

$$D_n = -\frac{a_n}{\gamma_n}\frac{k \operatorname{Sinh}\gamma_n h \operatorname{Cosh}\gamma_n h + \operatorname{Sinh}^2\gamma_n h}{k(\operatorname{Sinh}\gamma_n h \operatorname{Cosh}\gamma_n h + \gamma_n h) + \operatorname{Sinh}^2\gamma_n h - \gamma_n^2 h^2},$$

where $k = [E'(1 - \mu^2)]/[E(1 - \mu'^2)]$, the symbols E, μ and E', μ' denoting the modulus of elasticity and Poisson's ratio for the beam and the foundation respectively.

The general solution in (188 a–e) can also be employed for calculating the stress distribution in the foundation by using for the integration constants the modified values A'_n, B'_n, C'_n, and D'_n. We find that the pressure distribution on the surface of the foundation ($y_1 = 0$) will not be influenced by the last three of these constants, while for A'_n we have

$$A'_n = \frac{a_n}{\gamma_n^2}\frac{k(\operatorname{Sinh}\gamma_n h + \gamma_n h \operatorname{Cosh}\gamma_n h)}{k(\operatorname{Sinh}\gamma_n h \operatorname{Cosh}\gamma_n h + \gamma_n h) + \operatorname{Sinh}^2\gamma_n h - \gamma_n^2 h^2}.$$

By introducing the notation

$$c_n = \frac{k(\operatorname{Sinh}\gamma_n h + \gamma_n h \operatorname{Cosh}\gamma_n h)}{k(\operatorname{Sinh}\gamma_n h \operatorname{Cosh}\gamma_n h + \gamma_n h) + \operatorname{Sinh}^2\gamma_n h - \gamma_n^2 h^2}$$

we have from (188 a–c) the following solution for the pressure distribution $p(x)$, shearing force $Q(x)$, and bending moment $M(x)$ along the beam:

$$\left.\begin{aligned} p(x) &= -\left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n c_n \cos\gamma_n x\right), \\ Q(x) &= -\int_0^h \tau_{xy}\, dy = \sum_{n=1}^{\infty} \frac{a_n}{\gamma_n}(1 - c_n)\sin\gamma_n x, \\ M(x) &= \int_0^h \sigma_x y\, dy = \sum_{n=1}^{\infty} \frac{a_n}{\gamma_n^2}(1 - c_n)\cos\gamma_n x. \end{aligned}\right\} \qquad (189\text{ a–c})$$

For the axial force in the beam we have at the same time

$$N(x) = \int_0^h \sigma_x\, dy = 0.$$

Though the solution above is only for an infinitely long beam under distributed periodic loading, it has been shown by Prager* that from this case one can also obtain, under specified loading conditions, solutions for beams of finite length. For this purpose use is made of the particular properties of periodic load components which, when in series, are capable of reproducing a large variety of loading conditions.

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* *Op. cit.* (see p. 199).

The periodic load shown in Figure 156, for instance, can be developed in the form

$$q(x) = 2\alpha q_0 + \sum_{n=1}^{\infty} \frac{4q_0}{n\pi} \sin n\pi\alpha \cos n\pi\beta \cos \frac{n\pi x}{l}. \tag{190}$$

On account of the periodic symmetry of this loading the shearing force in the beam at $x = \pm l, \pm 3l, \pm 5l$, etc., will always be zero. By a suitable choice of the α and β ratios, however, one can also cancel the bending moments at those points, which will be equivalent to having beams with free ends.

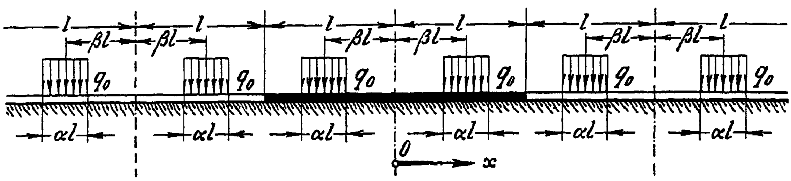

FIG. 156

Substituting $a_n = (4q_0/n\pi) \sin n\pi\alpha \cos n\pi\beta$ from (190) into (189c), we have for the bending moment at $x = l$

$$M_l = \frac{4q_0 l^2}{\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} (1 - c_n) \sin n\pi\alpha \cos n\pi\beta.$$

This equation can be used for the purpose of finding for any assumed value of α the corresponding β value which will make M_l vanish. Taking these values of α and β, we can write the distribution of foundation pressure and bending moment along each free-end beam of length $2l$ as

$$\left.\begin{aligned} p(x) &= 2\alpha q_0 + \frac{4q_0}{\pi} \sum_{n=1}^{\infty} \frac{c_n}{n} \sin n\pi\alpha \cos n\pi\beta \cos \frac{n\pi x}{l}, \\ M(x) &= \frac{4q_0 l^3}{\pi^3} \sum_{n=1}^{\infty} \frac{(1 - c_n)}{n^3} \sin n\pi\alpha \cos n\pi\beta \cos \frac{n\pi x}{l}. \end{aligned}\right\} \tag{191 a–b}$$

It should be noted that we got here the solution, not of the problem of a single beam of finite length supported on an elastic solid foundation, but of the problem in which the surface of the foundation is covered with an infinite row of individual beams, each of which has free ends and is loaded in the same manner.

56. *The Infinite Beam Supported on an Elastic Solid and Loaded by a Concentrated Force*

In the previous section we had the solution of the problem of an infinitely long beam supported on an elastic solid foundation and subjected along its upper edge to a distributed periodic loading. By superimposing periodic loading components one can also obtain a load equivalent to a concentrated force and thus derive the solution for the case when the infinite beam is under the action of a single concentrated load. This approach has been used by M. A. Biot,* whose results will be discussed below.

Consider a beam of width $2b$ supported on a foundation of the same width and subjected to a simple wave loading of the intensity

$$q = q_0 \cos \nu x.$$

The front and back faces of beam and foundation will be assumed to be free of surface tractions, and thus the problem under consideration will be that of plane stress. If the height of the beam is small compared to the wave length of the applied load, the flexural theory of beams can be used with good approximation, in which case we have for the deflection of the beam

$$w = \frac{q_0 \cos \nu x}{\nu E\left(1 + \frac{EI}{E'b}\nu^3\right)}, \tag{192a}$$

where EI is the flexural rigidity of the beam and E' is the elastic modulus of the foundation. At the same time the pressure distribution under the beam will be

$$p = \nu E'bw = \frac{q_0 \cos \nu x}{\frac{E}{E'b}\left(1 + \frac{EI}{E'b}\nu^3\right)}. \tag{192b}$$

Thus we find that in this particular loading there is a proportionality between pressure and deflection at every point along the beam, though the proportionality factor $k = \nu E'b$ is now not a constant but a function of the wave length of the loading. The bending moment in the beam due to the load $q = q_0 \cos \nu x$ is obtained as

$$M = \frac{q_0 \nu \cos \nu x}{\nu^3 + \frac{E'b}{EI}}. \tag{192c}$$

By superposing an infinite number of simple harmonic load components, like that considered above, any arbitrary $p(x)$ can be represented in the form of a Fourier integral:

$$p(x) = \frac{1}{\pi}\int_0^\infty d\nu \int_{-\infty}^{+\infty} p(\xi) \cos \nu(x - \xi)\, d\xi.$$

* "Bending of an Infinite Beam on an Elastic Foundation," Journal of Applied Mechanics, *Transactions of the American Society of Mechanical Engineers*, 59 (March, 1937), A 1-7.

Each element in this loading

$$\frac{1}{\pi}\, d\nu\, d\xi\, p(\xi) \cos \nu(x - \xi)$$

produces a bending moment which, according to (163c), is equivalent to

$$dM(x) = \frac{1}{\pi}\, d\nu\, d\xi\, \frac{\nu p(\xi) \cos \nu(x - \xi)}{\nu^3 + \dfrac{E'b}{EI}},$$

and consequently the total bending moment $M(x)$ due to the loading $p(x)$ is obtained as an integral of these elementary bending moments:

$$M(x) = \int_{-\infty}^{+\infty} p(\xi)\, d\xi \int_0^\infty \frac{d\nu}{\pi}\, \frac{\nu \cos \nu(x - \xi)}{\nu^3 + \dfrac{E'b}{EI}}.$$

If the loading is concentrated at $x = 0$ within a narrow region $x = \pm\epsilon$, we have

$$P = \int_{-\epsilon}^{+\epsilon} p(\xi)\, d\xi,$$

and the bending moment due to this concentrated load is obtained as

$$M(x) = \frac{P}{\pi} \int_0^\infty \frac{\nu \cos \nu x}{\nu^3 + \dfrac{E'b}{EI}}\, d\nu. \tag{193}$$

Introducing the symbol

$$\eta = \left(\frac{E'b}{EI}\right)^{\frac{1}{3}},$$

we can also express the bending moment above as a function of ηx:

$$M(x) = P \frac{1}{\eta}\, \varphi(\eta x),$$

where

$$\varphi(\eta x) = \frac{1}{\pi} \int_0^\infty \frac{\alpha \cos (\alpha \eta x)}{\alpha^3 + 1}\, d\alpha.$$

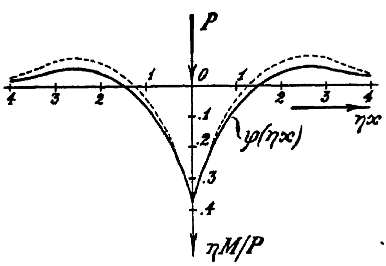

FIG. 157

This latter formula has been used by Biot in evaluating numerically the bending-moment distribution along the beam. He obtained the function $\varphi(\eta x)$, shown by the solid line in Figure 157, where the value of the maximum bending moment at the point of application of the load is

$$M_{\max} = 0.385\, \frac{P}{\eta}. \tag{194}$$

The bending moment in this case is proportional to the one-third power of the flexural rigidity of the beam and not to the one-fourth power found in (5c) for the Winkler type of foundation, where no continuity of the material was assumed. Equation (5c) gives for the maximum bending moment in the beam the same value as (194) if we take for the modulus of foundation k the value

$$k = 0.710E'\left(\frac{E'b^4}{EI}\right)^{\frac{1}{3}}. \tag{195}$$

The bending-moment curve calculated from (5c) with this value for k is shown by the dotted line in Figure 157.

The deflection lines corresponding to the bending moment curves above can be obtained in each case by integrating twice the formula

$$EI\frac{d^2w}{dx^2} = M(x).$$

The lines obtained in this manner for the elastic solid foundation on the one hand and for the Winkler foundation with the modulus k from (195) on the other hand are shown in Figure 158 by full and dotted lines respectively. Since in the elastic solid foundation the absolute value of the deflection is infinite everywhere, in order to compare the shape of these two types of deflection lines the ordinates at the point of application of the load were taken as the same.

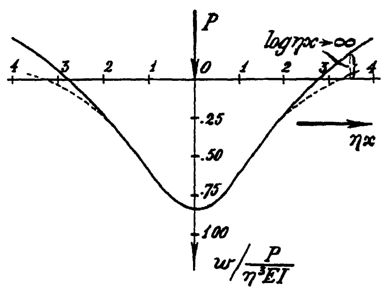

FIG. 158

The method above has also been applied by Biot* to the problem when the infinitely long beam of width $2b$ is supported on the surface of a three-dimensional elastic solid. In this case he found for the maximum bending moment in the beam under a concentrated load P the following value:

$$M_{\max} = 0.332Pb\left[C(1 - \mu'^2)\frac{EI}{E'b^4}\right]^{0.277}. \tag{196}$$

The value of the constant C in this equation depends on the character of distribution of pressure across the width $2b$ under the beam. For uniform distribution of pressure $C = 1$; whereas if the displacement in the foundation is constant across the width, $C = 1.13$. In either event the value of $M_{\max}$ from (196) will be very close to that obtained by the Winkler theory in two dimensions, which, by the simple substitution of $k = E'$ in (5c), gives

$$M_{\max} = 0.353P\left(\frac{EI}{E'}\right)^{0.250}.$$

* *Op. cit.* (see p. 205).

Thus we have the interesting conclusion that the simple assumption of a foundation without material continuity gives a closer approximation for the beam supported on a three-dimensional elastic solid, equation (196), than the corresponding two-dimensional case, equation (194).

In the derivation of these results the foundation was regarded as an elastic continuum, while the deformation of the beam was calculated on the basis of the elementary flexural theory. This method of approach has the advantage that the resulting formulas can be used for beams of any cross section, I-beams for instance, since the beams in this theory are defined merely by their flexural rigidity EI and the width of the base of their cross section $2b$. In consequence of these simplifying assumptions, however, the stress distribution in the beams has to be considered from a theoretical point of view as only approximate.

A rigorous solution for the two-dimensional problem of beams of rectangular cross section supported on an elastic solid foundation was obtained by Geoffrey Bosson,* who used stress and displacement functions for both beam and foundation and investigated also the case when the depth of the elastic foundation is finite. His method of solution will be discussed in detail below.

A rectangular coordinate system will be chosen where the x axis coincides with the interface between the beam and the foundation and the positive y axis points toward the beam, so that the upper and lower edges of the beam will be at $y = h$ and $y = 0$ respectively, while the rigid base of the elastic foundation will be at $y = -f$, as shown in Figure 159. The front and back faces of the beam and the foundation at $z = \pm c$ will be considered free of normal and shearing stresses, the problem being thus one of plane stress. It will be assumed that the loading along $y = h$ is purely normal and that there is no shearing stress along the planes $y = 0$ and $y = -f$.

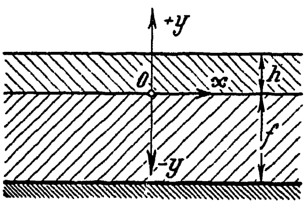

FIG. 159

In addition to the stress function F, which is biharmonic,

$$\Delta\Delta F = 0,$$

and the second derivatives of which give the normal and shearing stresses

$$\sigma_x = \frac{\partial^2 F}{\partial y^2}, \qquad \sigma_y = \frac{\partial^2 F}{\partial x^2}, \qquad \tau_{xy} = -\frac{\partial^2 F}{\partial x \, \partial y}, \tag{a}$$

there will be introduced a displacement function W of such a nature that the displacements u and v in the x and y directions are defined as

$$\left.\begin{aligned} Eu &= \frac{\partial W}{\partial y} - (1 + \mu)\frac{\partial F}{\partial x}, \\ Ev &= \frac{\partial W}{\partial x} - (1 + \mu)\frac{\partial F}{\partial y}, \end{aligned}\right\} \tag{b}$$

* "The Flexure of an Infinite Elastic Strip on an Elastic Foundation," *Philosophical Magazine*, Seventh Series, 27 (1939), 37–50.

E denoting the modulus of elasticity and μ representing Poisson's ratio for the material.

Between F and W we have the relation

$$\Delta F = \frac{\partial^2 W}{\partial x \, \partial y}, \quad \text{while} \quad \Delta W = 0.$$

The notations above will be used for the beam; when these symbols refer to the foundation they will be marked with a prime.

The first step in approaching the problem will be to establish the relationship between pressure and deflection at the surface of the foundation. This part of the investigation will involve only constants of the foundation and will not make any assumptions regarding the dimensions of the beam.

Assume for the foundation a stress function of the form

$$F' = \sum_{n=0}^{\infty} X_n \frac{y^n}{n!},$$

where X_n is a function of x only. Since the stress function has to satisfy the equation $\Delta\Delta F' = 0$, we find upon substitution of the expression above for F' that the result can be expressed in terms of four arbitrary functions of x, in the following symbolic form:*

$$F' = \left(\cos yD + \frac{1}{2} yD \sin yD\right) X_0 + \left(\frac{3}{2D} \sin yD - \frac{1}{2} y \cos yD\right) X_1$$
$$+ \frac{y}{2D} \cdot \sin yD \cdot X_2 + \frac{1}{2}\left(\frac{\sin yD}{D^3} - \frac{y}{D^2} \cos yD\right) X_3, \qquad (197a)$$

where

$$D \equiv \frac{d}{dx}.$$

The corresponding displacement function will be

$$W' = \sin yDX_0 - \frac{1}{D} \cos yDX_1 + \frac{1}{D^2} \sin yDX_2 - \frac{1}{D^3} \cos yDX_3. \qquad (197b)$$

By substituting the expressions above in (a) and (b) and denoting the modulus of elasticity of the material of the foundation by E', we find that

$$[\sigma_y]_{y=0} = D^2 X_0, \qquad E'[v]_{y=0} = -\frac{1}{D^2} X_3,$$

while $[\tau_{xy}]_{y=0} = -DX_1 = 0$ on account of the assumed boundary condition.

Since along the rigid base of the foundation we have $[\tau_{xy}]_{y=-f} = 0$ and

* Further information regarding stress functions of this type can be found in the paper by W. M. Shepherd, "On Generalized Plane Stress in a Thin Plane Plate," *Journal of the London Mathematical Society*, 4, Pt. 3 (1929), 213–219.

$[v]_{y=-f} = 0$, by introducing F' and W' into these conditions we obtain two simultaneous equations from which X_2 can be eliminated, giving

$$-2D\cdot\sin^2 fD\cdot X_0 = \left(\frac{1}{D^2}\sin fD\cdot\cos fD + \frac{f}{D}\right)X_3 .$$

Substituting here the expressions

$$X_0 = \frac{1}{D^2}[\sigma_y]_{y=0} \quad \text{and} \quad X_3 = -D^2E'[v]_{y=0},$$

which were obtained above, we have the fundamental relationship between pressure and deflection at the surface of the foundation:

$$2\sin^2 fD\cdot[\sigma_y]_{y=0} = \left(\frac{1}{D}\sin fD\cdot\cos fD + f\right)D^2E'[v]_{y=0} . \tag{198}$$

On the assumption that during deformation the beam remains in contact along its entire length with the foundation, the stress function for the beam will also have to satisfy the relationship obtained above in (198).

In the analysis of the beam we shall distinguish between two cases, according to whether the loading on the beam is an even or an odd function of x.

For the case when the loading is an even function of x the stress function will be assumed in the following form:

$$F_0 = (A\ \text{Cosh}\ my + By\ \text{Sinh}\ my + C\ \text{Sinh}\ my - Cmy\ \text{Cosh}\ my)\cos mx, \tag{c}$$

with the corresponding displacement function

$$W_0 = \frac{2}{m}(B\ \text{Sinh}\ my - Cm\ \text{Cosh}\ my)\sin mx. \tag{d}$$

The stress function in this form satisfies the boundary condition of zero shear at $y = 0$. The values of the three constants in the expressions above for F_0 and W_0 can be determined from the other conditions, namely, that the shear be zero at $y = h$ and that F_0 and W_0 give a pressure σ_y and a vertical displacement v along the interface $y = 0$ which will satisfy the relationship in (198). Carrying out these operations and assuming that the load along the upper edge $Q_0(x)$ is an even function of x, we have $Q_0(x) = -[\sigma_y]_{y=h}$, and the corresponding stress and displacement functions are obtained as

$$\left.\begin{aligned} F_0 &= \frac{2}{\pi}\int_0^\infty \frac{N_F}{m^2D_0}\cos mx\,dm\int_0^\infty Q_0(\xi)\cos m\xi\,d\xi, \\ W_0 &= \frac{4}{\pi}\int_0^\infty \frac{N_W}{m^2D_0}\sin mx\,dm\int_0^\infty Q_0(\xi)\cos m\xi\,d\xi, \end{aligned}\right\} \tag{199 a–b}$$

where

$$N_F = (\text{Sinh}\ mh + mh\ \text{Cosh}\ mh)\left[\left(\frac{E'}{E}\beta - \alpha my\right)\text{Cosh}\ my + \alpha\ \text{Sinh}\ my\right] + \left(mh\alpha - \frac{E'}{E}\beta\right)my\ \text{Sinh}\ mh\ \text{Sinh}\ my,$$

E is the elastic modulus of the material of the beam and E' that of the foundation,

$$N_W = \left(mh\alpha - \frac{E'}{E}\beta\right) \text{Sinh } mh \text{ Sinh } my - \alpha(\text{Sinh } mh + mh \text{ Cosh } mh) \text{ Cosh } my,$$

$$D_0 = \frac{E'}{E}\beta(\text{Sinh } mh \text{ Cosh } mh + mh) + \alpha(\text{Sinh}^2 mh - m^2h^2),$$

and

$$\alpha = 2 \text{ Sinh}^2 mf, \qquad \beta = \text{Sinh } 2mf + 2mf.$$

For the case when the loading on the beam is an odd function of x, $Q_1(x)$, the stress function will be assumed in the form $F_1 = F(y) \sin mx$, $F(y)$ denoting the same expression as that in parentheses on the right side of (c). In a manner similar to that used above, the stress function F_1 and the displacement function W_1 are obtained now as

$$\left.\begin{aligned} F_1 &= \frac{2}{\pi}\int_0^\infty \frac{N_F}{m^2 D_0} \sin mx \, dm \int_0^\infty Q_1(\xi) \sin m\xi \, d\xi, \\ W_1 &= -\frac{4}{\pi}\int_0^\infty \frac{N_W}{m^2 D_0} \cos mx \, dm \int_0^\infty Q_1(\xi) \sin m\xi \, d\xi, \end{aligned}\right\} \qquad (200\text{ a–b})$$

where N_F, N_W, and D_0 denote the same expressions as in (199 a–b).

If the depth of the foundation is large, the expressions (199 a–b) and (200 a–b) can be simplified by putting $\alpha/\beta = 1$.

By the combination of these two cases of even and odd distributed loadings solutions can be derived for any arbitrary load $Q(x)$ on the beam. Since

$$Q(x) = Q_0(x) + Q_1(x) \qquad \text{and} \qquad Q(-x) = Q_0(x) - Q_1(x),$$

the even and odd components of $Q(x)$ are obtained as

$$Q_0(x) = \tfrac{1}{2}[Q(x) + Q(-x)] \qquad \text{and} \qquad Q_1(x) = \tfrac{1}{2}[Q(x) - Q(-x)].$$

The stress and displacement functions corresponding to these components can be obtained from (199 a–b) and (200 a–b) respectively, and thus we have for the general type of loading $Q(x)$

$$F = F_0 + F_1$$

and

$$W = W_0 + W_1.$$

Let us consider now the particular problem when the beam is under the action of a concentrated force P at $x = 0$, $y = h$, while the depth of the foundation is large ($\alpha = \beta$) and the beam and the foundation are of the same material ($E = E'$).

Since the loading in this case is

$$\int_0^\infty Q_0(\xi) \cos m\xi \, d\xi = \frac{P}{2},$$

the corresponding stress and displacement functions are obtained as

$$\left.\begin{aligned} F_0 &= \frac{P}{\pi} \int_0^\infty \frac{N_F}{m^2 D_0} \cos mx \, dm, \\ W_0 &= \frac{2P}{\pi} \int_0^\infty \frac{N_W}{m^2 D_0} \sin mx \, dm, \end{aligned}\right\} \qquad (201\text{ a–b})$$

where N_F, N_W, and D_0 denote the same expressions as in (199 a–b) except that they are now simplified by having $\alpha = \beta$ and $E = E'$. For the stress components in the beam we have now the following formulas:

$$\begin{aligned} \sigma_x = \frac{P}{\pi} \int_0^\infty \frac{1}{D_0} \{&(\text{Sinh}\, mh + mh \,\text{Cosh}\, mh)[(1 + my) \,\text{Cosh}\, my - \text{Sinh}\, my] \\ &+ (mh - 1) \,\text{Sinh}\, mh(2 \,\text{Cosh}\, my + my \,\text{Sinh}\, my)\} \cos mx \, dm, \end{aligned} \qquad (202\text{a})$$

$$\begin{aligned} \sigma_y = -\frac{P}{\pi} \int_0^\infty \frac{1}{D_0} \{&(\text{Sinh}\, mh + mh \,\text{Cosh}\, mh)[(1 - my) \,\text{Cosh}\, my + \text{Sinh}\, my] \\ &+ (mh - 1)my \,\text{Sinh}\, mh \,\text{Sinh}\, my\} \cos mx \, dm, \end{aligned} \qquad (202\text{b})$$

$$\begin{aligned} \tau_{xy} = \frac{P}{\pi} \int_0^\infty \frac{1}{D_0} [&(\text{Sinh}\, mh + mh \,\text{Cosh}\, mh)(1 - my) \,\text{Sinh}\, my \\ &+ (mh - 1) \,\text{Sinh}\, mh(\text{Sinh}\, my + my \,\text{Cosh}\, my)] \sin mx \, dm. \end{aligned} \qquad (202\text{c})$$

The pressure distribution and deflection along the interface between beam and foundation are obtained as

$$\left.\begin{aligned} [\sigma_y]_{y=0} &= -\frac{P}{\pi} \int_0^\infty \frac{(\text{Sinh}\, mh + mh \,\text{Cosh}\, mh) \cos mx \, dm}{\text{Sinh}\, mh \,\text{Cosh}\, mh + \text{Sinh}^2 mh + mh - m^2 h^2}, \\ E[v]_{y=0} &= -\frac{2P}{\pi} \int_0^\infty \frac{(\text{Sinh}\, mh + mh \,\text{Cosh}\, mh) \cos mx \, dm}{m(\text{Sinh}\, mh \,\text{Cosh}\, mh + \text{Sinh}^2 mh + mh - m^2 h^2)}. \end{aligned}\right\} \qquad (203\text{ a–b})$$

Bosson* has evaluated the integrals above numerically, partly by means of Filon's quadrature formulas† and partly by asymptotic expansion. The expression in (203b) is a divergent integral, which can, however, be replaced by a convergent one by assuming zero deflection for the point of application of the load and referring the rest of the deflection ordinates to this point as origin. Denoting the vertical displacement components so interpreted by $\bar{v}$, Bosson has ob-

* *Op. cit.* (see p. 208).

† L. N. G. Filon, "On a Quadrature Formula for Trigonometric Integrals," *Proceedings of the Royal Society of Edinburgh*, 49 (1929), 38–47.

tained the following values of pressure and deflection along the beam in terms of the x/h ratios:

$\frac{x}{h}$	0	± 0.5	± 1.0	± 2.0	± 3.0
$\frac{\pi h}{P}[\sigma_y]_{y=0}$	-2.1709	-1.4095	-0.5548	-0.0248	0.0244
$\frac{\pi E}{4P}[\bar{v}]_{y=0}$	0	0.3376	0.9330	1.7947	–

A graphic representation of these curves is shown in Figures 160 a and b. It is seen that the pressure on the interface is zero at $x/h = 2.25$ and becomes negative (tension) for larger values of x/h. The area of the pressure diagram within the region $x/h = \pm 2.25$ is equivalent to $-1.01P$, that is, only 1 per cent larger than necessary to balance the load. This shows that the solution above would be quite accurate even for the case when no tension could be transmitted through the interface between beam and foundation.

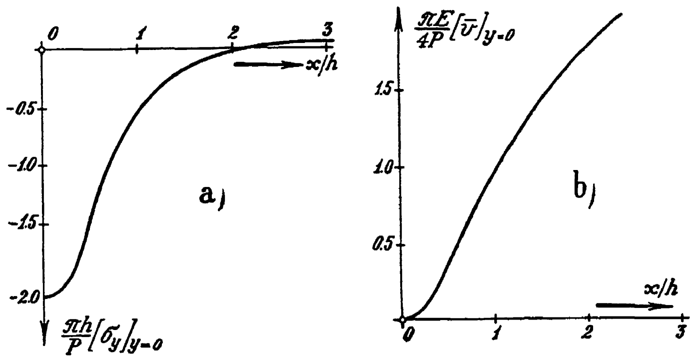

Fig. 160

From the numerical values in the accompanying table one can also calculate the ratio between pressure and deflection at various points along the interface. The relationship, shown in Figure 161, turns out not to be linear, proving that it would be incorrect to assume a proportionality between pressure and deflection at every point along the beam in such a case when the foundation of the beam is an elastic solid of unlimited depth.

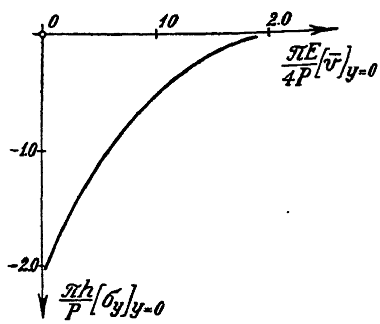

Fig. 161

From the analysis above interesting conclusions can be drawn also for the problem when the depth of the foundation f is small compared to the height of the beam h. Expanding the operators in (198) according to ascending powers of f and retaining only the first terms, we have in this case

$$fD^2[\sigma_y]_{y=0} = D^2E'[v]_{y=0}\,.$$

Introducing here, as in the flexural theory, the notation y for the deflection of the beam, we obtain the following differential equation for the deflection line:

$$\left(D^4 + \frac{cE'}{f}\frac{1}{EI}\right)D^2y = 0.$$

By taking for the modulus of the foundation $k = cE'/f$, where c is the width, f is the depth, and E' is the modulus of elasticity of the foundation, we find that the solution of the equation above is, except for a linear term, the same as that obtained in (3), where no continuity in the material of the foundation was assumed. Hence it is seen that, if the foundation under the beam is of a continuous elastic material but of small depth, a correct solution can be obtained on the basis of the simple assumption that the pressure in the foundation is proportional at every point to the deflection of the beam at that point.

TABLES

FORMULAS

Trigonometric Functions

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$$\cos x \pm i \sin x = e^{\pm ix}$$

$$\cos^2 x + \sin^2 x = 1$$

$$\cos^2 x - \sin^2 x = \cos 2x$$

$$2 \sin x \cos x = \sin 2x$$

$$\sin^2 \frac{x}{2} = \frac{1}{2}(1 - \cos x)$$

$$\cos^2 \frac{x}{2} = \frac{1}{2}(1 + \cos x)$$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}$$

$$\sin x - \sin y = 2 \cos \frac{x+y}{2} \sin \frac{x-y}{2}$$

$$\cos x + \cos y = 2 \cos \frac{x+y}{2} \cos \frac{x-y}{2}$$

$$\cos x - \cos y = -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2}$$

Hyperbolic Functions

$$\text{Sinh}\, x = \frac{e^{x} - e^{-x}}{2}$$

$$\text{Cosh}\, x = \frac{e^{x} + e^{-x}}{2}$$

$$\text{Cosh}\, x \pm \text{Sinh}\, x = e^{\pm x}$$

$$\text{Cosh}^2 x - \text{Sinh}^2 x = 1$$

$$\text{Cosh}^2 x + \text{Sinh}^2 x = \text{Cosh}\, 2x$$

$$2\, \text{Sinh}\, x\, \text{Cosh}\, x = \text{Sinh}\, 2x$$

$$\text{Sinh}^2 \frac{x}{2} = \frac{1}{2}(\text{Cosh}\, x - 1)$$

$$\text{Cosh}^2 \frac{x}{2} = \frac{1}{2}(\text{Cosh}\, x + 1)$$

$$\text{Sinh}(x \pm y) = \text{Sinh}\, x\, \text{Cosh}\, y \pm \text{Cosh}\, x\, \text{Sinh}\, y$$

$$\text{Cosh}(x \pm y) = \text{Cosh}\, x\, \text{Cosh}\, y \pm \text{Sinh}\, x\, \text{Sinh}\, y$$

$$\text{Sinh}\, x + \text{Sinh}\, y = 2\, \text{Sinh} \frac{x+y}{2} \text{Cosh} \frac{x-y}{2}$$

$$\text{Sinh}\, x - \text{Sinh}\, y = 2\, \text{Cosh} \frac{x+y}{2} \text{Sinh} \frac{x-y}{2}$$

$$\text{Cosh}\, x + \text{Cosh}\, y = 2\, \text{Cosh} \frac{x+y}{2} \text{Cosh} \frac{x-y}{2}$$

$$\text{Cosh}\, x - \text{Cosh}\, y = 2\, \text{Sinh} \frac{x+y}{2} \text{Sinh} \frac{x-y}{2}$$

1 radian = 57°17′44.806″

1° = 0.01745329 rad., 1′ = 0.00029089 rad., 1″ = 0.00000485 rad.

TABLE I

$\sin x$, $\cos x$, $\operatorname{Sinh} x$, $\operatorname{Cosh} x$, e^x, e^{-x},

$A_x = e^{-x}(\cos x + \sin x)$, $B_x = e^{-x}\sin x$,

$C_x = e^{-x}(\cos x - \sin x)$, $D_x = e^{-x}\cos x$.

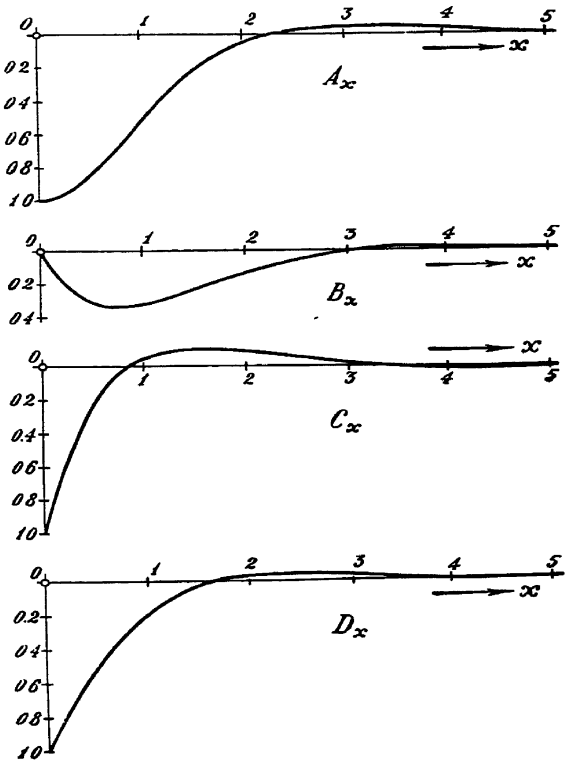

FIG. 162

TABLE I

x	sin x	cos x	Sinh x	Cosh x	e^x
0	0	1	0	1	1
0.001	0.0010	1.0000	0.0010	1.0000	1.0010
0.002	0.0020	1.0000	0.0020	1.0000	1.0020
0.003	0.0030	1.0000	0.0030	1.0000	1.0030
0.004	0.0040	1.0000	0.0040	1.0000	1.0040
0.005	0.0050	1.0000	0.0050	1.0000	1.0050
0.006	0.0060	1.0000	0.0060	1.0000	1.0060
0.007	0.0070	1.0000	0.0070	1.0000	1.0070
0.008	0.0080	1.0000	0.0080	1.0000	1.0080
0.009	0.0090	1.0000	0.0090	1.0000	1.0090
0.010	0.0100	1.0000	0.0100	1.0000	1.0100
0.011	0.0110	0.9999	0.0110	1.0001	1.0111
0.012	0.0120	0.9999	0.0120	1.0001	1.0121
0.013	0.0130	0.9999	0.0130	1.0001	1.0131
0.014	0.0140	0.9999	0.0140	1.0001	1.0141
0.015	0.0150	0.9999	0.0150	1.0001	1.0151
0.016	0.0160	0.9999	0.0160	1.0001	1.0161
0.017	0.0170	0.9999	0.0170	1.0001	1.0172
0.018	0.0180	0.9998	0.0180	1.0002	1.0182
0.019	0.0190	0.9998	0.0190	1.0002	1.0192
0.02	0.0200	0.9998	0.0200	1.0002	1.0202
0.03	0.0300	0.9996	0.0300	1.0004	1.0304
0.04	0.0400	0.9992	0.0400	1.0008	1.0408
0.05	0.0500	0.9988	0.0500	1.0012	1.0513
0.06	0.0600	0.9982	0.0600	1.0018	1.0618
0.07	0.0699	0.9976	0.0701	1.0024	1.0725
0.08	0.0799	0.9968	0.0801	1.0032	1.0833
0.09	0.0899	0.9960	0.0901	1.0040	1.0942
0.10	0.0998	0.9950	0.1002	1.0050	1.1052
0.11	0.1098	0.9940	0.1102	1.0061	1.1163
0.12	0.1197	0.9928	0.1203	1.0072	1.1275
0.13	0.1296	0.9916	0.1304	1.0085	1.1388
0.14	0.1395	0.9902	0.1405	1.0098	1.1503
0.15	0.1494	0.9888	0.1506	1.0113	1.1618
0.16	0.1593	0.9872	0.1607	1.0128	1.1735
0.17	0.1692	0.9856	0.1708	1.0145	1.1853
0.18	0.1790	0.9838	0.1810	1.0162	1.1972
0.19	0.1889	0.9820	0.1912	1.0181	1.2092
0.20	0.1987	0.9801	0.2013	1.0201	1.2214
0.21	0.2085	0.9780	0.2116	1.0221	1.2337
0.22	0.2182	0.9759	0.2218	1.0243	1.2461
0.23	0.2280	0.9737	0.2320	1.0266	1.2586
0.24	0.2377	0.9713	0.2423	1.0289	1.2712

TABLE I

x	e^{-x}	A_x	B_x	C_x	D_x
0	1	1	0	1	1
0.001	0.9990	1.0000	0.0010	0.9980	0.9990
0.002	0.9980	1.0000	0.0020	0.9960	0.9980
0.003	0.9970	1.0000	0.0030	0.9940	0.9970
0.004	0.9960	1.0000	0.0040	0.9920	0.9960
0.005	0.9950	1.0000	0.0050	0.9900	0.9950
0.006	0.9940	1.0000	0.0060	0.9880	0.9940
0.007	0.9930	0.9999	0.0070	0.9861	0.9930
0.008	0.9920	0.9999	0.0080	0.9841	0.9920
0.009	0.9910	0.9999	0.0087	0.9821	0.9910
0.010	0.9900	0.9999	0.0099	0.9801	0.9900
0.011	0.9891	0.9999	0.0109	0.9781	0.9890
0.012	0.9881	0.9999	0.0119	0.9761	0.9880
0.013	0.9871	0.9998	0.0129	0.9742	0.9870
0.014	0.9861	0.9998	0.0138	0.9722	0.9860
0.015	0.9851	0.9998	0.0148	0.9702	0.9850
0.016	0.9841	0.9997	0.0158	0.9683	0.9840
0.017	0.9831	0.9997	0.0167	0.9663	0.9830
0.018	0.9822	0.9997	0.0177	0.9643	0.9820
0.019	0.9812	0.9996	0.0187	0.9624	0.9810
0.02	0.9802	0.9996	0.0196	0.9604	0.9800
0.03	0.9704	0.9991	0.0291	0.9409	0.9700
0.04	0.9608	0.9984	0.0384	0.9216	0.9600
0.05	0.9512	0.9976	0.0476	0.9025	0.9501
0.06	0.9418	0.9966	0.0565	0.8836	0.9401
0.07	0.9324	0.9954	0.0653	0.8649	0.9302
0.08	0.9231	0.9940	0.0738	0.8464	0.9202
0.09	0.9139	0.9924	0.0822	0.8281	0.9103
0.10	0.9048	0.9906	0.0903	0.8100	0.9003
0.11	0.8958	0.9887	0.0983	0.7921	0.8904
0.12	0.8869	0.9867	0.1062	0.7744	0.8806
0.13	0.8781	0.9844	0.1138	0.7568	0.8707
0.14	0.8694	0.9821	0.1213	0.7395	0.8608
0.15	0.8607	0.9796	0.1286	0.7224	0.8510
0.16	0.8521	0.9770	0.1358	0.7055	0.8413
0.17	0.8437	0.9742	0.1427	0.6888	0.8315
0.18	0.8353	0.9713	0.1495	0.6722	0.8218
0.19	0.8270	0.9683	0.1562	0.6550	0.8121
0.20	0.8187	0.9651	0.1627	0.6398	0.8024
0.21	0.8106	0.9618	0.1690	0.6238	0.7928
0.22	0.8025	0.9583	0.1752	0.6080	0.7832
0.23	0.7945	0.9547	0.1812	0.5924	0.7736
0.24	0.7866	0.9511	0.1870	0.5771	0.7641

TABLE I

x	sin x	cos x	Sinh x	Cosh x	e^x
0.25	0.2474	0.9689	0.2526	1.0314	1.2840
0.26	0.2571	0.9664	0.2629	1.0340	1.2969
0.27	0.2667	0.9638	0.2733	1.0367	1.3100
0.28	0.2764	0.9611	0.2837	1.0395	1.3231
0.29	0.2860	0.9582	0.2941	1.0424	1.3364
0.30	0.2955	0.9553	0.3045	1.0453	1.3500
0.31	0.3051	0.9523	0.3150	1.0484	1.3634
0.32	0.3146	0.9492	0.3255	1.0516	1.3771
0.33	0.3240	0.9460	0.3360	1.0550	1.3910
0.34	0.3335	0.9428	0.3466	1.0584	1.4050
0.35	0.3429	0.9394	0.3572	1.0619	1.4191
0.36	0.3523	0.9359	0.3678	1.0655	1.4333
0.37	0.3616	0.9323	0.3785	1.0692	1.4477
0.38	0.3709	0.9287	0.3892	1.0731	1.4623
0.39	0.3802	0.9249	0.4000	1.0770	1.4770
0.40	0.3894	0.9211	0.4108	1.0811	1.4918
0.41	0.3986	0.9171	0.4216	1.0852	1.5068
0.42	0.4078	0.9131	0.4325	1.0895	1.5220
0.43	0.4169	0.9090	0.4434	1.0939	1.5373
0.44	0.4259	0.9048	0.4543	1.0984	1.5527
0.45	0.4350	0.9004	0.4653	1.1030	1.5683
0.46	0.4440	0.8960	0.4764	1.1077	1.5841
0.47	0.4529	0.8916	0.4875	1.1125	1.6000
0.48	0.4618	0.8870	0.4986	1.1174	1.6161
0.49	0.4706	0.8823	0.5098	1.1225	1.6323
0.50	0.4794	0.8776	0.5211	1.1276	1.6487
0.51	0.4882	0.8727	0.5324	1.1329	1.6653
0.52	0.4969	0.8678	0.5438	1.1383	1.6820
0.53	0.5055	0.8628	0.5552	1.1438	1.6989
0.54	0.5141	0.8577	0.5666	1.1494	1.7160
0.55	0.5227	0.8525	0.5782	1.1551	1.7332
0.56	0.5312	0.8473	0.5897	1.1609	1.7507
0.57	0.5396	0.8419	0.6014	1.1669	1.7683
0.58	0.5480	0.8365	0.6131	1.1730	1.7860
0.59	0.5564	0.8309	0.6248	1.1792	1.8040
0.60	0.5646	0.8253	0.6366	1.1855	1.8221
0.61	0.5729	0.8196	0.6485	1.1919	1.8404
0.62	0.5810	0.8139	0.6605	1.1984	1.8589
0.63	0.5891	0.8080	0.6725	1.2051	1.8776
0.64	0.5972	0.8021	0.6846	1.2119	1.8965
0.65	0.6052	0.7961	0.6968	1.2188	1.9155
0.66	0.6131	0.7900	0.7090	1.2258	1.9348
0.67	0.6210	0.7838	0.7213	1.2330	1.9542
0.68	0.6288	0.7776	0.7336	1.2402	1.9739
0.69	0.6365	0.7712	0.7461	1.2476	1.9937

TABLE I

x	e^{-x}	A_x	B_x	C_x	D_x
0.25	0.7788	0.9472	0.1927	0.5619	0.7546
0.26	0.7710	0.9433	0.1982	0.5469	0.7451
0.27	0.7634	0.9393	0.2036	0.5321	0.7357
0.28	0.7558	0.9353	0.2089	0.5175	0.7264
0.29	0.7483	0.9310	0.2140	0.5030	0.7171
0.30	0.7408	0.9267	0.2189	0.4888	0.7078
0.31	0.7334	0.9222	0.2237	0.4748	0.6985
0.32	0.7262	0.9177	0.2284	0.4609	0.6893
0.33	0.7189	0.9130	0.2330	0.4472	0.6801
0.34	0.7118	0.9084	0.2374	0.4337	0.6710
0.35	0.7047	0.9036	0.2416	0.4204	0.6620
0.36	0.6977	0.8986	0.2457	0.4072	0.6530
0.37	0.6907	0.8938	0.2497	0.3943	0.6440
0.38	0.6839	0.8887	0.2536	0.3815	0.6351
0.39	0.6771	0.8836	0.2574	0.3688	0.6262
0.40	0.6703	0.8784	0.2610	0.3564	0.6174
0.41	0.6636	0.8732	0.2646	0.3441	0.6087
0.42	0.6570	0.8679	0.2680	0.3320	0.6000
0.43	0.6505	0.8625	0.2712	0.3201	0.5913
0.44	0.6440	0.8570	0.2743	0.3084	0.5827
0.45	0.6376	0.8515	0.2774	0.2968	0.5742
0.46	0.6313	0.8459	0.2803	0.2853	0.5657
0.47	0.6250	0.8403	0.2832	0.2742	0.5573
0.48	0.6188	0.8346	0.2857	0.2632	0.5489
0.49	0.6126	0.8289	0.2883	0.2522	0.5406
0.50	0.6065	0.8231	0.2908	0.2414	0.5323
0.51	0.6005	0.8173	0.2932	0.2307	0.5241
0.52	0.5945	0.8113	0.2954	0.2204	0.5159
0.53	0.5886	0.8054	0.2976	0.2103	0.5079
0.54	0.5828	0.7994	0.2996	0.2002	0.4998
0.55	0.5770	0.7934	0.3016	0.1902	0.4918
0.56	0.5712	0.7873	0.3035	0.1805	0.4839
0.57	0.5655	0.7813	0.3052	0.1709	0.4761
0.58	0.5599	0.7752	0.3068	0.1615	0.4683
0 59	0.5543	0.7690	0.3084	0.1522	0.4606
0.60	0.5488	0.7628	0.3099	0.1430	0.4529
0.61	0.5434	0.7566	0.3113	0.1340	0.4453
0.62	0.5379	0.7503	0.3126	0.1252	0.4378
0.63	0.5326	0.7442	0.3138	0.1166	0.4301
0.64	0.5273	0.7379	0.3150	0.1080	0.4230
0.65	0.5220	0.7315	0.3160	0.0996	0.4156
0.66	0.5168	0.7252	0.3169	0.0914	0.4083
0.67	0.5117	0.7189	0.3178	0.0833	0.4011
0.68	0.5066	0.7126	0.3186	0.0754	0.3940
0.69	0.5016	0.7062	0.3193	0.0676	0.3869

TABLE I

x	$\sin x$	$\cos x$	$\text{Sinh}\ x$	$\text{Cosh}\ x$	e^x
0.70	0.6442	0.7648	0.7586	1.2552	2.0138
0.71	0.6518	0.7584	0.7712	1.2628	2.0340
0.72	0.6594	0.7518	0.7838	1.2706	2.0544
0.73	0.6669	0.7452	0.7966	1.2785	2.0751
0.74	0.6743	0.7385	0.8094	1.2865	2.0959
0.75	0.6816	0.7317	0.8223	1.2947	2.1170
0.76	0.6889	0.7248	0.8353	1.3030	2.1383
0.77	0.6961	0.7179	0.8484	1.3114	2.1598
0.78	0.7033	0.7109	0.8615	1.3199	2.1815
$\frac{1}{4}\pi$	0.7071	0.7071	0.8687	1.3246	2.1933
0.79	0.7104	0.7038	0.8748	1.3286	2.2034
0.80	0.7174	0.6967	0.8881	1.3374	2.2255
0.81	0.7243	0.6895	0.9015	1.3464	2.2479
0.82	0.7312	0.6822	0.9150	1.3555	2.2705
0.83	0.7379	0.6749	0.9286	1.3647	2.2933
0.84	0.7446	0.6675	0.9423	1.3740	2.3164
0.85	0.7513	0.6600	0.9561	1.3835	2.3396
0.86	0.7578	0.6524	0.9700	1.3932	2.3632
0.87	0.7643	0.6448	0.9840	1.4029	2.3869
0.88	0.7707	0.6372	0.9981	1.4128	2.4109
0.89	0.7771	0.6294	1.0122	1.4229	2.4351
0.90	0.7833	0.6216	1.0265	1.4331	2.4596
0.91	0.7895	0.6138	1.0409	1.4434	2.4843
0.92	0.7956	0.6058	1.0554	1.4539	2.5093
0.93	0.8016	0.5978	1.0700	1.4645	2.5345
0.94	0.8076	0.5898	1.0847	1.4753	2.5600
0.95	0.8134	0.5817	1.0995	1.4862	2.5857
0.96	0.8192	0.5735	1.1144	1.4973	2.6117
0.97	0.8249	0.5653	1.1294	1.5085	2.6379
0.98	0.8305	0.5570	1.1446	1.5200	2.6645
0.99	0.8360	0.5487	1.1598	1.5314	2.6912
1.00	0.8415	0.5403	1.1752	1.5431	2.7183
1.01	0.8468	0.5319	1.1907	1.5549	2.7456
1.02	0.8521	0.5234	1.2063	1.5669	2.7732
1.03	0.8573	0.5148	1.2220	1.5790	2.8011
1.04	0.8624	0.5062	1.2379	1.5913	2.8292
1.05	0.8674	0.4976	1.2539	1.6038	2.8576
1.06	0.8724	0.4889	1.2700	1.6164	2.8864
1.07	0.8772	0.4801	1.2862	1.6292	2.9154
1.08	0.8820	0.4713	1.3025	1.6421	2.9447
1.09	0.8866	0.4625	1.3190	1.6552	2.9743
1.10	0.8912	0.4536	1.3356	1.6685	3.0042
1.11	0.8957	0.4447	1.3524	1.6820	3.0344
1.12	0.9001	0.4357	1.3693	1.6956	3.0648
1.13	0.9044	0.4267	1.3863	1.7093	3.0957
1.14	0.9086	0.4176	1.4035	1.7233	3.1268

TABLE I

x	e^{-x}	A_x	B_x	C_x	D_x
0.70	0.4966	0.6997	0.3199	0.0599	0.3798
0.71	0.4916	0.6933	0.3205	0.0524	0.3729
0.72	0.4868	0.6869	0.3210	0.0449	0.3659
0.73	0.4819	0.6805	0.3214	0.0377	0.3591
0.74	0.4771	0.6741	0.3217	0.0307	0.3524
0.75	0.4724	0.6676	0.3220	0.0237	0.3456
0.76	0.4677	0.6611	0.3221	0.0168	0.3389
0.77	0.4630	0.6547	0.3223	0.0101	0.3324
0.78	0.4584	0.6483	0.3224	0.0035	0.3259
$\frac{1}{4}\pi$	0.4559	0.6448	0.3224	0	0.3224
0.79	0.4538	0.6418	0.3224	−0.0030	0.3195
0.80	0.4493	0.6353	0.3223	−0.0093	0.3131
0.81	0.4449	0.6289	0.3222	−0.0155	0.3067
0.82	0.4404	0.6225	0.3221	−0.0217	0.3004
0.83	0.4360	0.6160	0.3219	−0.0276	0.2943
0.84	0.4317	0.6096	0.3215	−0.0334	0.2881
0.85	0.4274	0.6032	0.3212	−0.0391	0.2821
0.86	0.4232	0.5968	0.3207	−0.0446	0.2761
0.87	0.4190	0.5904	0.3202	−0.0500	0.2702
0.88	0.4148	0.5840	0.3197	−0.0554	0.2643
0.89	0.4107	0.5776	0.3191	−0.0606	0.2585
0.90	0.4066	0.5712	0.3185	−0.0658	0.2527
0.91	0.4025	0.5648	0.3178	−0.0708	0.2470
0.92	0.3985	0.5584	0.3171	−0.0757	0.2414
0.93	0.3946	0.5521	0.3169	−0.0805	0.2359
0.94	0.3906	0.5459	0.3155	−0.0851	0.2304
0.95	0.3867	0.5396	0.3146	−0.0896	0.2250
0.96	0.3829	0.5333	0.3107	−0.0941	0.2196
0.97	0.3791	0.5270	0.3127	−0.0984	0.2143
0.98	0.3753	0.5207	0.3117	−0.1027	0.2090
0.99	0.3716	0.5145	0.3107	−0.1069	0.2038
1.00	0.3679	0.5083	0.3096	−0.1109	0.1987
1.01	0.3642	0.5021	0.3085	−0.1147	0.1937
1.02	0.3606	0.4960	0.3073	−0.1185	0.1888
1.03	0.3570	0.4899	0.3061	−0.1223	0.1839
1.04	0.3534	0.4839	0.3049	−0.1259	0.1790
1.05	0.3499	0.4778	0.3036	−0.1294	0.1742
1.06	0.3465	0.4716	0.3023	−0.1328	0.1694
1.07	0.3430	0.4656	0.3009	−0.1362	0.1647
1.08	0.3396	0.4596	0.2995	−0.1394	0.1601
1.09	0.3362	0.4536	0.2981	−0.1426	0.1555
1.10	0.3329	0.4476	0.2967	−0.1458	0.1509
1.11	0.3296	0.4416	0.2952	−0.1488	0.1464
1.12	0.3263	0.4356	0.2936	−0.1516	0.1420
1.13	0.3230	0.4298	0.2921	−0.1543	0.1378
1.14	0.3198	0.4240	0.2906	−0.1570	0.1335

TABLE I

x	sin x	cos x	Sinh x	Cosh x	e^x
1.15	0.9128	0.4085	1.4208	1.7374	3.1582
1.16	0.9168	0.3993	1.4382	1.7517	3.1899
1.17	0.9208	0.3902	1.4558	1.7662	3.2220
1.18	0.9246	0.3809	1.4736	1.7808	3.2544
1.19	0.9284	0.3717	1.4914	1.7956	3.2871
1.20	0.9320	0.3624	1.5095	1.8107	3.3201
1.21	0.9356	0.3530	1.5276	1.8258	3.3535
1.22	0.9391	0.3436	1.5460	1.8412	3.3872
1.23	0.9425	0.3342	1.5645	1.8568	3.4212
1.24	0.9458	0.3248	1.5831	1.8725	3.4556
1.25	0.9490	0.3153	1.6019	1.8884	3.4903
1.26	0.9521	0.3058	1.6209	1.9045	3.5254
1.27	0.9551	0.2963	1.6400	1.9208	3.5608
1.28	0.9580	0.2867	1.6593	1.9373	3.5966
1.29	0.9608	0.2771	1.6788	1.9540	3.6328
1.30	0.9636	0.2675	1.6984	1.9709	3.6693
1.31	0.9662	0.2578	1.7182	1.9880	3.7062
1.32	0.9687	0.2482	1.7381	2.0053	3.7434
1.33	0.9712	0.2385	1.7583	2.0228	3.7810
1.34	0.9735	0.2288	1.7786	2.0404	3.8190
1.35	0.9757	0.2190	1.7991	2.0583	3.8574
1.36	0.9779	0.2092	1.8198	2.0764	3.8962
1.37	0.9799	0.1994	1.8406	2.0947	3.9354
1.38	0.9818	0.1896	1.8617	2.1132	3.9749
1.39	0.9837	0.1798	1.8829	2.1320	4.0148
1.40	0.9854	0.1700	1.9043	2.1509	4.0552
1.41	0.9871	0.1601	1.9259	2.1700	4.0960
1.42	0.9886	0.1502	1.9477	2.1894	4.1371
1.43	0.9901	0.1403	1.9697	2.2090	4.1787
1.44	0.9915	0.1304	1.9919	2.2288	4.2207
1.45	0.9927	0.1205	2.0143	2.2488	4.2631
1.46	0.9939	0.1106	2.0369	2.2691	4.3060
1.47	0.9949	0.1006	2.0596	2.2896	4.3492
1.48	0.9959	0.0907	2.0826	2.3103	4.3930
1.49	0.9967	0.0807	2.1059	2.3312	4.4371
1.50	0.9975	0.0707	2.1293	2.3524	4.4817
1.51	0.9982	0.0608	2.1529	2.3738	4.5267
1.52	0.9987	0.0508	2.1768	2.3955	4.5722
1.53	0.9992	0.0408	2.2008	2.4174	4.6182
1.54	0.9995	0.0308	2.2251	2.4395	4.6646
1.55	0.9998	0.0208	2.2496	2.4619	4.7115
1.56	0.9999	0.0108	2.2743	2.4845	4.7588
1.57	1.0000	0.0008	2.2993	2.5074	4.8066
$\frac{1}{2}\pi$	1	0	2.3013	2.5092	4.8105
1.58	1.0000	−0.0092	2.3245	2.5305	4.8550
1.59	0.9998	−0.0192	2.3499	2.5538	4.9038

TABLE I

x	e^{-x}	A_x	B_x	C_x	D_x
1.15	0.3166	0.4183	0.2890	−0.1597	0.1293
1.16	0.3135	0.4126	0.2874	−0.1622	0.1252
1.17	0.3104	0.4069	0.2858	−0.1647	0.1241
1.18	0.3073	0.4012	0.2842	−0.1671	0.1171
1.19	0.3042	0.3955	0.2825	−0.1694	0.1131
1.20	0.3012	0.3898	0.2807	−0.1716	0.1091
1.21	0.2982	0.3842	0.2790	−0.1737	0.1053
1.22	0.2952	0.3786	0.2773	−0.1758	0.1014
1.23	0.2923	0.3731	0.2755	−0.1778	0.0977
1.24	0.2894	0.3677	0.2737	−0.1797	0.0940
1.25	0.2865	0.3623	0.2719	−0.1815	0.0904
1.26	0.2836	0.3569	0.2701	−0.1833	0.0868
1.27	0.2808	0.3515	0.2683	−0.1849	0.0833
1.28	0.2780	0.3462	0.2664	−0.1865	0.0798
1.29	0.2753	0.3408	0.2645	−0.1881	0.0763
1.30	0.2725	0.3355	0.2626	−0.1897	0.0729
1.31	0.2698	0.3303	0.2607	−0.1911	0.0696
1.32	0.2671	0.3251	0.2588	−0.1925	0.0663
1.33	0.2645	0.3199	0.2569	−0.1938	0.0631
1.34	0.2618	0.3148	0.2550	−0.1950	0.0600
1.35	0.2592	0.3098	0.2530	−0.1962	0.0568
1.36	0.2567	0.3047	0.2510	−0.1973	0.0537
1.37	0.2541	0.2997	0.2490	−0.1983	0.0507
1.38	0.2516	0.2948	0.2470	−0.1993	0.0478
1.39	0.2491	0.2898	0.2450	−0.2003	0.0448
1.40	0.2466	0.2849	0.2430	−0.2011	0.0419
1.41	0.2441	0.2801	0.2410	−0.2019	0.0391
1.42	0.2417	0.2753	0.2390	−0.2027	0.0363
1.43	0.2393	0.2705	0.2370	−0.2033	0.0336
1.44	0.2369	0.2658	0.2349	−0.2039	0.0309
1.45	0.2346	0.2611	0.2329	−0.2045	0.0283
1.46	0.2322	0.2565	0.2308	−0.2051	0.0257
1.47	0.2299	0.2519	0.2288	−0.2056	0.0232
1.48	0.2276	0.2474	0.2267	−0.2060	0.0207
1.49	0.2254	0.2429	0.2247	−0.2064	0.0183
1.50	0.2231	0.2384	0.2226	−0.2068	0.0158
1.51	0.2209	0.2339	0.2205	−0.2071	0.0134
1.52	0.2187	0.2295	0.2184	−0.2073	0.0111
1.53	0.2165	0.2252	0.2164	−0.2075	0.0089
1.54	0.2144	0.2209	0.2143	−0.2077	0.0066
1.55	0.2122	0.2166	0.2122	−0.2078	0.0044
1.56	0.2101	0.2123	0.2101	−0.2079	0.0022
1.57	0.2080	0.2082	0.2081	−0.2079	0.0002
$\frac{1}{2}\pi$	0.2079	0.2079	0.2079	−0.2079	0
1.58	0.2060	0.2041	0.2060	−0.2079	−0.0019
1.59	0.2039	0.2000	0.2039	−0.2078	−0.0039

TABLE I

x	sin x	cos x	Sinh x	Cosh x	e^x
1.60	0.9996	−0.0292	2.3756	2.5775	4.9530
1.61	0.9992	−0.0392	2.4015	2.6014	5.0028
1.62	0.9988	−0.0492	2.4276	2.6255	5.0531
1.63	0.9982	−0.0592	2.4540	2.6499	5.1039
1.64	0.9976	−0.0692	2.4806	2.6746	5.1552
1.65	0.9969	−0.0791	2.5075	2.6995	5.2070
1.66	0.9960	−0.0891	2.5346	2.7247	5.2593
1.67	0.9951	−0.0990	2.5620	2.7502	5.3122
1.68	0.9940	−0.1090	2.5896	2.7760	5.3656
1.69	0.9929	−0.1189	2.6175	2.8020	5.4195
1.70	0.9917	−0.1288	2.6456	2.8283	5.4740
1.71	0.9903	−0.1388	2.6740	2.8549	5.5290
1.72	0.9889	−0.1486	2.7027	2.8818	5.5845
1.73	0.9874	−0.1585	2.7317	2.9090	5.6406
1.74	0.9857	−0.1684	2.7609	2.9364	5.6973
1.75	0.9840	−0.1782	2.7904	2.9642	5.7546
1.76	0.9822	−0.1881	2.8202	2.9922	5.8124
1.77	0.9802	−0.1979	2.8503	3.0206	5.8708
1.78	0.9782	−0.2077	2.8806	3.0492	5.9299
1.79	0.9761	−0.2174	2.9112	3.0782	5.9894
1.80	0.9738	−0.2272	2.9422	3.1075	6.0496
1.81	0.9715	−0.2369	2.9734	3.1370	6.1104
1.82	0.9691	−0.2466	3.0049	3.1669	6.1719
1.83	0.9666	−0.2563	3.0367	3.1972	6.2339
1.84	0.9640	−0.2660	3.0689	3.2277	6.2965
1.85	0.9613	−0.2756	3.1013	3.2585	6.3598
1.86	0.9585	−0.2852	3.1340	3.2897	6.4237
1.87	0.9556	−0.2948	3.1671	3.3212	6.4883
1.88	0.9526	−0.3043	3.2005	3.3530	6.5535
1.89	0.9495	−0.3138	3.2342	3.3852	6.6194
1.90	0.9463	−0.3233	3.2682	3.4177	6.6859
1.91	0.9430	−0.3327	3.3025	3.4506	6.7531
1.92	0.9396	−0.3422	3.3372	3.4838	6.8210
1.93	0.9362	−0.3515	3.3722	3.5173	6.8895
1.94	0.9326	−0.3609	3.4075	3.5512	6.9588
1.95	0.9290	−0.3702	3.4432	3.5855	7.0287
1.96	0.9252	−0.3794	3.4792	3.6201	7.0993
1.97	0.9214	−0.3887	3.5156	3.6551	7.1707
1.98	0.9174	−0.3979	3.5523	3.6904	7.2427
1.99	0.9134	−0.4070	3.5894	3.7261	7.3155
2.00	0.9093	−0.4162	3.6269	3.7622	7.3891
2.01	0.9051	−0.4252	3.6647	3.7986	7.4633
2.02	0.9008	−0.4342	3.7028	3.8355	7.5383
2.03	0.8964	−0.4432	3.7414	3.8727	7.6141
2.04	0.8919	−0.4522	3.7803	3.9103	7.6906

TABLE I

x	e^{-x}	A_x	B_x	C_x	D_x
1.60	0.2019	0.1960	0.2018	−0.2077	−0.0059
1.61	0.1999	0.1919	0.1997	−0.2075	−0.0078
1.62	0.1979	0.1879	0.1976	−0.2073	−0.0097
1.63	0.1959	0.1840	0.1956	−0.2071	−0.0116
1.64	0.1940	0.1801	0.1935	−0.2069	−0.0134
1.65	0.1920	0.1763	0.1915	−0.2067	−0.0152
1.66	0.1901	0.1725	0.1894	−0.2064	−0.0170
1.67	0.1882	0.1686	0.1873	−0.2060	−0.0187
1.68	0.1864	0.1648	0.1852	−0.2056	−0.0204
1.69	0.1845	0.1612	0.1832	−0.2051	−0.0220
1.70	0.1827	0.1576	0.1812	−0.2046	−0.0236
1.71	0.1809	0.1540	0.1791	−0.2042	−0.0251
1.72	0.1791	0.1505	0.1771	−0.2037	−0.0266
1.73	0.1773	0.1470	0.1751	−0.2032	−0.0281
1.74	0.1755	0.1435	0.1730	−0.2026	−0.0296
1.75	0.1738	0.1400	0.1720	−0.2020	−0.0310
1.76	0.1720	0.1365	0.1690	−0.2013	−0.0324
1.77	0.1703	0.1332	0.1670	−0.2006	−0.0338
1.78	0.1686	0.1299	0.1650	−0.2000	−0.0351
1.79	0.1670	0.1266	0.1630	−0.1993	−0.0364
1.80	0.1653	0.1234	0.1610	−0.1985	−0.0376
1.81	0.1636	0.1202	0.1590	−0.1978	−0.0388
1.82	0.1620	0.1170	0.1570	−0.1970	−0.0400
1.83	0.1604	0.1138	0.1550	−0.1962	−0.0412
1.84	0.1588	0.1108	0.1531	−0.1953	−0.0423
1.85	0.1572	0.1078	0.1512	−0.1945	−0.0434
1.86	0.1557	0.1048	0.1492	−0.1936	−0.0444
1.87	0.1541	0.1018	0.1473	−0.1927	−0.0454
1.88	0.1526	0.0989	0.1453	−0.1917	−0.0464
1.89	0.1511	0.0960	0.1434	−0.1908	−0.0474
1.90	0.1496	0.0932	0.1415	−0.1899	−0.0484
1.91	0.1481	0.0904	0.1396	−0.1889	−0.0493
1.92	0.1466	0.0876	0.1377	−0.1879	−0.0501
1.93	0.1452	0.0849	0.1359	−0.1869	−0.0510
1.94	0.1437	0.0822	0.1340	−0.1859	−0.0519
1.95	0.1423	0.0795	0.1322	−0.1849	−0.0527
1.96	0.1409	0.0769	0.1304	−0.1838	−0.0535
1.97	0.1395	0.0743	0.1285	−0.1827	−0.0543
1.98	0.1381	0.0717	0.1267	−0.1816	−0.0550
1.99	0.1367	0.0692	0.1249	−0.1804	−0.0556
2.00	0.1353	0.0667	0.1230	−0.1793	−0.0563
2.01	0.1340	0.0643	0.1213	−0.1782	−0.0569
2.02	0.1327	0.0619	0.1195	−0.1771	−0.0576
2.03	0.1313	0.0595	0.1128	−0.1759	−0.0582
2.04	0.1300	0.0571	0.1160	−0.1748	−0.0588

TABLE I

x	sin x	cos x	Sinh x	Cosh x	e^x
2.05	0.8874	−0.4611	3.8196	3.9483	7.7679
2.06	0.8827	−0.4699	3.8593	3.9867	7.8460
2.07	0.8780	−0.4787	3.8993	4.0255	7.9248
2.08	0.8731	−0.4875	3.9398	4.0647	8.0045
2.09	0.8682	−0.4962	3.9806	4.1043	8.0849
2.10	0.8632	−0.5048	4.0219	4.1443	8.1662
2.11	0.8581	−0.5134	4.0635	4.1847	8.2482
2.12	0.8529	−0.5220	4.1056	4.2256	8.3311
2.13	0.8477	−0.5305	4.1480	4.2668	8.4149
2.14	0.8423	−0.5390	4.1909	4.3086	8.4994
2.15	0.8369	−0.5474	4.2342	4.3507	8.5849
2.16	0.8314	−0.5557	4.2779	4.3932	8.6711
2.17	0.8258	−0.5640	4.3220	4.4362	8.7583
2.18	0.8201	−0.5722	4.3666	4.4797	8.8463
2.19	0.8143	−0.5804	4.4116	4.5236	8.9352
2.20	0.8085	−0.5885	4.4571	4.5679	9.0250
2.21	0.8026	−0.5966	4.5030	4.6127	9.1157
2.22	0.7966	−0.6046	4.5494	4.6580	9.2073
2.23	0.7905	−0.6125	4.5962	4.7037	9.2999
2.24	0.7843	−0.6204	4.6434	4.7499	9.3933
2.25	0.7781	−0.6282	4.6912	4.7966	9.4877
2.26	0.7718	−0.6359	4.7394	4.8437	9.5831
2.27	0.7654	−0.6436	4.7880	4.8914	9.6794
2.28	0.7589	−0.6512	4.8372	4.9395	9.7767
2.29	0.7523	−0.6588	4.8868	4.9881	9.8749
2.30	0.7457	−0.6663	4.9370	5.0372	9.9742
2.31	0.7390	−0.6737	4.9876	5.0868	10.0744
2.32	0.7322	−0.6811	5.0387	5.1370	10.1757
2.33	0.7254	−0.6883	5.0903	5.1876	10.2779
2.34	0.7185	−0.6956	5.1424	5.2388	10.3812
2.35	0.7115	−0.7027	5.1951	5.2905	10.4856
$\frac{3}{4}\pi$	0.7071	−0.7071	5.2280	5.3228	10.5507
2.36	0.7044	−0.7098	5.2483	5.3427	10.5910
2.37	0.6973	−0.7168	5.3020	5.3954	10.6974
2.38	0.6901	−0.7237	5.3562	5.4487	10.8049
2.39	0.6828	−0.7306	5.4109	5.5026	10.9135
2.40	0.6755	−0.7374	5.4662	5.5570	11.0232
2.41	0.6681	−0.7441	5.5221	5.6119	11.1340
2.42	0.6606	−0.7508	5.5785	5.6674	11.2459
2.43	0.6530	−0.7573	5.6354	5.7235	11.3589
2.44	0.6454	−0.7638	5.6929	5.7801	11.4730
2.45	0.6378	−0.7702	5.7510	5.8373	11.5884
2.46	0.6300	−0.7766	5.8097	5.8951	11.7048
2.47	0.6222	−0.7828	5.8689	5.9535	11.8224
2.48	0.6144	−0.7890	5.9288	6.0125	11.9413
2.49	0.6064	−0.7951	5.9892	6.0721	12.0613

TABLE I

x	sin x	cos x	Sinh x	Cosh x	e^x
2.50	0.5985	−0.8011	6.0502	6.1323	12.1825
2.51	0.5904	−0.8071	6.1118	6.1931	12.3049
2.52	0.5823	−0.8130	6.1741	6.2545	12.4286
2.53	0.5742	−0.8187	6.2369	6.3166	12.5535
2.54	0.5660	−0.8244	6.3004	6.3793	12.6797
2.55	0.5577	−0.8300	6.3645	6.4426	12.8071
2.56	0.5494	−0.8356	6.4293	6.5066	12.9358
2.57	0.5410	−0.8410	6.4946	6.5712	13.0658
2.58	0.5325	−0.8464	6.5607	6.6365	13.1971
2.59	0.5240	−0.8517	6.6274	6.7024	13.3298
2.60	0.5155	−0.8569	6.6947	6.7690	13.4637
2.61	0.5069	−0.8620	6.7628	6.8363	13.5990
2.62	0.4983	−0.8670	6.8315	6.9043	13.7357
2.63	0.4896	−0.8720	6.9008	6.9729	13.8738
2.64	0.4808	−0.8768	6.9709	7.0423	14.0132
2.65	0.4720	−0.8816	7.0417	7.1123	14.1540
2.66	0.4632	−0.8863	7.1132	7.1831	14.2963
2.67	0.4543	−0.8908	7.1854	7.2546	14.4400
2.68	0.4454	−0.8953	7.2583	7.3268	14.5851
2.69	0.4364	−0.8998	7.3319	7.3998	14.7317
2.70	0.4274	−0.9041	7.4063	7.4735	14.8797
2.71	0.4183	−0.9083	7.4814	7.5479	15.0293
2.72	0.4092	−0.9124	7.5572	7.6231	15.1803
2.73	0.4001	−0.9165	7.6338	7.6990	15.3329
2.74	0.3909	−0.9204	7.7112	7.7758	15.4870
2.75	0.3817	−0.9243	7.7894	7.8533	15.6426
2.76	0.3724	−0.9281	7.8683	7.9316	15.7998
2.77	0.3631	−0.9318	7.9480	8.0106	15.9586
2.78	0.3538	−0.9353	8.0285	8.0905	16.1190
2.79	0.3444	−0.9388	8.1098	8.1712	16.2810
2.80	0.3350	−0.9422	8.1919	8.2527	16.4446
2.81	0.3256	−0.9455	8.2749	8.3351	16.6099
2.82	0.3161	−0.9487	8.3586	8.4182	16.7768
2.83	0.3066	−0.9518	8.4432	8.5022	16.9455
2.84	0.2970	−0.9549	8.5287	8.5871	17.1158
2.85	0.2875	−0.9578	8.6150	8.6728	17.2878
2.86	0.2779	−0.9606	8.7021	8.7594	17.4615
2.87	0.2683	−0.9633	8.7902	8.8469	17.6370
2.88	0.2586	−0.9660	8.8791	8.9352	17.8143
2.89	0.2490	−0.9685	8.9689	9.0244	17.9933
2.90	0.2392	−0.9710	9.0596	9.1146	18.1742
2.91	0.2295	−0.9733	9.1512	9.2056	18.3568
2.92	0.2198	−0.9756	9.2437	9.2976	18.5413
2.93	0.2100	−0.9777	9.3371	9.3905	18.7276
2.94	0.2002	−0.9798	9.4315	9.4844	18.9158

TABLE I

e^{-x}	A_x	B_x	C_x	D_x
0.0821	−0.0166	0.0492	−0.1149	−0.0658
0.0813	−0.0176	0.0480	−0.1136	−0.0656
0.0805	−0.0185	0.0464	−0.1123	−0.0654
0.0797	−0.0195	0.0457	−0.1109	−0.0652
0.0789	−0.0204	0.0446	−0.1096	−0.0650
0.0781	−0.0213	0.0435	−0.1083	−0.0648
0.0773	−0.0221	0.0425	−0.1071	−0.0646
0.0765	−0.0228	0.0414	−0.1058	−0.0644
0.0758	−0.0237	0.0403	−0.1045	−0.0642
0.0750	−0.0246	0.0394	−0.1033	−0.0640
0.0743	−0.0254	0.0383	−0.1020	−0.0637
0.0735	−0.0261	0.0373	−0.1007	−0.0634
0.0728	−0.0269	0.0363	−0.0994	−0.0632
0.0721	−0.0276	0.0353	−0.0982	−0.0629
0.0714	−0.0283	0.0343	−0.0969	−0.0626
0.0706	−0.0289	0.0334	−0.0956	−0.0623
0.0700	−0.0296	0.0324	−0.0944	−0.0620
0.0692	−0.0302	0.0315	−0.0932	−0.0617
0.0686	−0.0308	0.0306	−0.0920	−0.0614
0.0679	−0.0314	0.0297	−0.0908	−0.0611
0.0672	−0.0320	0.0287	−0.0895	−0.0608
0.0665	−0.0326	0.0279	−0.0883	−0.0605
0.0659	−0.0331	0.0270	−0.0871	−0.0601
0.0652	−0.0337	0.0261	−0.0859	−0.0598
0.0646	−0.0342	0.0253	−0.0847	−0.0594
0.0639	−0.0347	0.0244	−0.0835	−0.0591
0.0633	−0.0352	0.0236	−0.0823	−0.0588
0.0627	−0.0356	0.0228	−0.0811	−0.0585
0.0620	−0.0361	0.0220	−0.0799	−0.0581
0.0614	−0.0365	0.0212	−0.0787	−0.0577
0.0608	−0.0369	0.0204	−0.0777	−0.0573
0.0602	−0.0373	0.0196	−0.0765	−0.0570
0.0596	−0.0377	0.0188	−0.0754	−0.0566
0.0590	−0.0381	0.0181	−0.0742	−0.0562
0.0584	−0.0385	0.0173	−0.0731	−0.0558
0.0578	−0.0388	0.0167	−0.0721	−0.0554
0.0573	−0.0391	0.0160	−0.0710	−0.0550
0.0567	−0.0394	0.0153	−0.0699	−0.0546
0.0561	−0.0397	0.0145	−0.0687	−0.0542
0.0556	−0.0400	0.0138	−0.0676	−0.0538
0.0550	−0.0403	0.0132	−0.0666	−0.0534
0.0545	−0.0406	0.0125	−0.0656	−0.0530
0.0539	−0.0409	0.0114	−0.0645	−0.0526
0.0534	−0.0411	0.0112	−0.0634	−0.0522
0.0529	−0.0413	0.0106	−0.0624	−0.0518

TABLE I

x	sin x	cos x	Sinh x	Cosh x	e^x
2.95	0.1904	−0.9817	9.5268	9.5792	19.1060
2.96	0.1806	−0.9836	9.6231	9.6749	19.2980
2.97	0.1708	−0.9853	9.7203	9.7716	19.4919
2.98	0.1609	−0.9870	9.8185	9.8693	19.6878
2.99	0.1510	−0.9885	9.9177	9.9680	19.8857
3.00	0.1411	−0.9900	10.0179	10.0677	20.0855
3.01	0.1312	−0.9914	10.1190	10.1684	20.2874
3.02	0.1213	−0.9926	10.2212	10.2700	20.4913
3.03	0.1114	−0.9938	10.3245	10.3728	20.6972
3.04	0.1014	−0.9948	10.4287	10.4765	20.9052
3.05	0.0915	−0.9958	10.5340	10.5814	21.1153
3.06	0.0815	−0.9967	10.6403	10.6872	21.3276
3.07	0.0715	−0.9974	10.7477	10.7942	21.5419
3.08	0.0616	−0.9981	10.8562	10.9022	21.7584
3.09	0.0516	−0.9987	10.9658	10.0113	21.9771
3.10	0.0416	−0.9991	11.0764	11.1215	22.1980
3.11	0.0316	−0.9995	11.1882	11.2328	22.4210
3.12	0.0216	−0.9998	11.3011	11.3453	22.6464
3.13	0.0116	−0.9999	11.4151	11.4588	22.8740
3.14	0.0016	−1.0000	11.5303	11.5736	23.1039
π	0	−1	11.5487	11.5920	23.1407
3.15	−0.0084	−1.0000	11.6466	11.6895	23.3361
3.16	−0.0184	−0.9998	11.7641	11.8065	23.5706
3.17	−0.0284	−0.9996	11.8827	11.9247	23.8075
3.18	−0.0384	−0.9993	12.0026	12.0442	24.0468
3.19	−0.0484	−0.9988	12.1236	12.1648	24.2884
3.20	−0.0584	−0.9983	12.2459	12.2866	24.5325
3.21	−0.0684	−0.9977	12.3694	12.4097	24.7791
3.22	−0.0783	−0.9969	12.4941	12.5340	25.0281
3.23	−0.0883	−0.9961	12.6200	12.6596	25.2797
3.24	−0.0982	−0.9952	12.7473	12.7864	25.5337
3.25	−0.1082	−0.9941	12.8758	12.9146	25.7903
3.26	−0.1181	−0.9930	13.0056	13.0440	26.0495
3.27	−0.1280	−0.9918	13.1367	13.1747	26.3113
3.28	−0.1380	−0.9904	13.2691	13.3067	26.5758
3.29	−0.1479	−0.9890	13.4028	13.4401	26.8429
3.30	−0.1578	−0.9875	13.5379	13.5748	27.1126
3.31	−0.1676	−0.9858	13.6743	13.7108	27.3851
3.32	−0.1775	−0.9841	13.8121	13.8482	27.6604
3.33	−0.1873	−0.9823	13.9513	13.9871	27.9383
3.34	−0.1971	−0.9804	14.0918	14.1273	28.2191
3.35	−0.2069	−0.9784	14.2338	14.2689	28.5027
3.36	−0.2169	−0.9762	14.3772	14.4120	28.7892
3.37	−0.2264	−0.9740	14.5221	14.5565	29.0785
3.38	−0.2362	−0.9717	14.6684	14.7024	29.3708
3.39	−0.2459	−0.9693	14.8161	14.8498	29.6660

TABLE I

e^{-x}	A_x	B_x	C_x	D_x
0.0523	−0.0415	0.0100	−0.0614	−0.0514
0.0518	−0.0417	0.0094	−0 0603	−0.0510
0.0513	−0.0419	0.0088	−0.0593	−0.0506
0.0508	−0.0420	0.0082	−0.0583	−0.0502
0.0503	−0.0421	0.0076	−0.0573	−0.0497
0.0498	−0.0422	0.0071	−0.0563	−0.0493
0.0493	−0.0423	0.0065	−0.0553	−0.0489
0.0488	−0.0424	0.0059	−0.0543	−0.0484
0.0483	−0.0425	0.0054	−0.0534	−0.0480
0.0478	−0.0426	0.0049	−0.0524	−0.0476
0.0474	−0.0427	0.0043	−0.0515	−0.0472
0.0469	−0.0428	0.0039	−0.0505	−0.0468
0.0464	−0.0429	0.0034	−0.0496	−0.0464
0.0460	−0.0430	0.0029	−0.0487	−0.0459
0.0455	−0.0431	0.0023	−0.0478	−0.0455
0.0450	−0.0431	0.0019	−0.0469	−0.0450
0.0446	−0.0431	0.0015	−0.0460	−0.0446
0.0442	−0.0432	0.0010	−0.0451	−0.0441
0.0437	−0.0432	0.0006	−0.0442	−0.0437
0.0433	−0.0432	0.0001	−0.0433	−0.0432
0.0432	−0.0432	0	−0.0432	−0.0432
0.0428	−0.0432	−0.0004	−0.0424	−0.0428
0.0424	−0.0432	−0.0008	−0.0416	−0.0423
0.0420	−0.0432	−0.0012	−0.0407	−0.0420
0.0416	−0.0431	−0.0016	−0.0399	−0.0415
0.0412	−0.0431	−0.0020	−0.0391	−0.0411
0.0408	−0.0431	−0.0024	−0.0383	−0.0407
0.0404	−0.0430	−0.0029	−0.0375	−0.0403
0.0400	−0.0430	−0.0032	−0.0367	−0.0399
0.0396	−0.0429	−0.0035	−0.0359	−0.0394
0.0392	−0.0428	−0.0039	−0.0351	−0.0390
0.0388	−0.0427	−0.0042	−0.0343	−0.0385
0.0384	−0.0426	−0.0046	−0.0336	−0.0381
0.0380	−0.0425	−0.0049	−0.0328	−0.0377
0.0376	−0.0424	−0.0052	−0.0321	−0.0373
0.0372	−0.0423	−0.0055	−0.0313	−0.0369
0.0369	−0.0422	−0.0058	−0.0306	−0.0365
0.0365	−0.0421	−0.0061	−0.0299	−0.0360
0.0362	−0.0420	−0.0064	−0.0292	−0.0356
0.0358	−0.0419	−0.0067	−0.0285	−0.0352
0.0354	−0.0418	−0.0070	−0.0278	−0.0398
0.0351	−0.0417	−0.0073	−0.0271	−0.0344
0.0347	−0.0415	−0.0075	−0.0264	−0.0340
0.0344	−0.0413	−0.0078	−0.0257	−0.0335
0.0340	−0.0411	−0.0080	−0.0251	−0.0331
0.0337	−0.0409	−0.0083	−0.0244	−0.0327

TABLE I

x	sin x	cos x	Sinh x	Cosh x	e^x
3.40	−0.2555	−0.9668	14.9654	14.9987	29.9641
3.41	−0.2652	−0.9642	15.1161	15.1491	30.2652
3.42	−0.2748	−0.9615	15.2684	15.3011	30.5694
3.43	−0.2844	−0.9587	15.4221	15.4545	30.8766
3.44	−0.2940	−0.9558	15.5774	15.6095	31.1870
3.45	−0.3035	−0.9528	15.7343	15.7661	31.5004
3.46	−0.3130	−0.9497	15.8928	15.9242	31.8170
3.47	−0.3225	−0.9466	16.0528	16.0839	32.1367
3.48	−0.3320	−0.9433	16.2145	16.2453	32.4597
3.49	−0.3414	−0.9399	16.3777	16.4082	32.7860
3.50	−0.3508	−0.9365	16.5426	16.5728	33.1154
3.51	−0.3601	−0.9329	16.7092	16.7391	33.4483
3.52	−0.3694	−0.9292	16.8774	16.9070	33.7844
3.53	−0.3787	−0.9255	17.0473	17.0766	34.1240
3.54	−0.3880	−0.9217	17.2190	17.2480	34.4669
3.55	−0.3972	−0.9178	17.3923	17.4210	34.8133
3.56	−0.4063	−0.9137	17.5674	17.5958	35.1632
3.57	−0.4154	−0.9096	17.7442	17.7724	35.5166
3.58	−0.4245	−0.9054	17.9228	17.9507	35.8735
3.59	−0.4335	−0.9011	18.1032	18.1308	36.2341
3.60	−0.4425	−0.8968	18.2855	18.3128	36.5982
3 61	−0.4515	−0.8923	18.4695	18.4966	36.9660
3.62	−0.4604	−0.8877	18.6554	18.6822	37.3376
3.63	−0.4692	−0.8831	18.8432	18.8697	37.7128
3.64	−0.4780	−0.8784	19.0328	19.0590	38.0918
3.65	−0.4868	−0.8735	19.2243	19.2503	38.4747
3.66	−0.4955	−0.8686	19.4178	19.4435	38.8613
3.67	−0.5042	−0.8636	19.6132	19.6387	39.2519
3.68	−0.5128	−0.8585	19.8106	19.8358	39.6464
3.69	−0.5213	−0.8534	20.0099	20.0349	40.0448
3.70	−0.5298	−0.8481	20.2113	20.2360	40.4473
3.71	−0.5383	−0.8428	20.4147	20.4391	40.8538
3.72	−0.5467	−0.8373	20.6201	20.6443	41.2644
3.73	−0.5550	−0.8318	20.8276	20.8516	41.6791
3.74	−0.5633	−0.8262	21.0371	21.0609	42.0980
3.75	−0.5716	−0.8206	21.2488	21.2723	42.5211
3.76	−0.5797	−0.8148	21.4626	21.4858	42.9484
3.77	−0.5879	−0.8090	21.6785	21.7016	43.3801
3.78	−0.5959	−0.8030	21.8966	21.9194	43.8160
3.79	−0.6039	−0.7970	22.1169	22.1395	44.2564
3.80	−0.6119	−0.7910	22.3394	22.3618	44.7012
3.81	−0.6197	−0.7848	22.5642	22.5863	45.1504
3.82	−0.6276	−0.7786	22.7911	22.8131	45.6042
3.83	−0.6353	−0.7723	23.0204	23.0421	46.0625
3.84	−0.6430	−0.7659	23.2520	23.2735	46.5255

TABLE I

e^{-x}	A_x	B_x	C_x	D_x
0.0334	−0.0408	−0.0085	−0.0238	−0.0323
0.0330	−0.0406	−0.0088	−0.0231	−0.0319
0.0327	−0.0404	−0.0090	−0.0225	−0.0315
0.0324	−0.0403	−0.0093	−0.0218	−0.0311
0.0321	−0.0401	−0.0094	−0.0212	−0.0307
0.0318	−0.0399	−0.0097	−0.0206	−0.0303
0.0314	−0.0397	−0.0099	−0.0200	−0.0299
0.0311	−0.0395	−0.0101	−0.0194	−0.0295
0.0308	−0.0392	−0.0102	−0.0189	−0.0291
0.0305	−0.0390	−0.0104	−0.0183	−0.0287
0.0302	−0.0388	−0.0106	−0.0177	−0.0283
0.0299	−0.0386	−0.0108	−0.0171	−0.0279
0.0296	−0.0384	−0.0109	−0.0165	−0.0275
0.0293	−0.0382	−0.0111	−0.0160	−0.0271
0.0290	−0.0380	−0.0113	−0.0155	−0.0268
0.0287	−0.0378	−0.0114	−0.0149	−0.0264
0.0284	−0.0376	−0.0116	−0.0144	−0.0260
0.0282	−0.0373	−0.0117	−0.0139	−0.0257
0.0279	−0.0371	−0.0118	−0.0134	−0.0253
0.0276	−0.0368	−0.0120	−0.0129	−0.0249
0.0273	−0.0366	−0.0121	−0.0124	−0.0245
0.0270	−0.0363	−0.0122	−0.0119	−0.0242
0.0268	−0.0361	−0.0123	−0.0114	−0.0238
0.0265	−0.0359	−0.0124	−0.0109	−0.0234
0.0262	−0.0356	−0.0125	−0.0105	−0.0231
0.0260	−0.0354	−0.0126	−0.0101	−0.0227
0.0257	−0.0351	−0.0127	−0.0096	−0.0223
0.0255	−0.0348	−0.0128	−0.0092	−0.0220
0.0252	−0.0346	−0.0129	−0.0088	−0.0217
0.0250	−0.0343	−0.0130	−0.0083	−0.0214
0.0247	−0.0341	−0.0131	−0.0079	−0.0210
0.0245	−0.0338	−0.0132	−0.0075	−0.0207
0.0242	−0.0336	−0.0132	−0.0071	−0.0203
0.0240	−0.0333	−0.0133	−0.0067	−0.0200
0.0238	−0.0330	−0.0133	−0.0063	−0.0197
0.0235	−0.0327	−0.0134	−0.0059	−0.0193
0.0233	−0.0324	−0.0135	−0.0055	−0.0190
0.0230	−0.0322	−0.0136	−0.0051	−0.0187
0.0228	−0.0319	−0.0136	−0.0048	−0.0184
0.0226	−0.0316	−0.0137	−0.0044	−0.0180
0.0224	−0.0314	−0.0137	−0.0040	−0.0177
0.0222	−0.0311	−0.0138	−0.0036	−0.0174
0.0219	−0.0308	−0.0138	−0.0033	−0.0171
0.0217	−0.0305	−0.0138	−0.0030	−0.0168
0.0215	−0.0303	−0.0138	−0.0027	−0.0165

TABLE I

x	$\sin x$	$\cos x$	$\operatorname{Sinh} x$	$\operatorname{Cosh} x$	e^x
3.85	−0.6506	−0.7594	23.4859	23.5072	46.9931
3.86	−0.6582	−0.7528	23.7221	23.7432	47.4654
3.87	−0.6657	−0.7462	23.9608	23.9816	47.9424
3.88	−0.6731	−0.7395	24.2018	24.2224	48.4242
3.89	−0.6805	−0.7328	24.4452	24.4657	48.9109
3.90	−0.6878	−0.7259	24.6911	24.7114	49.4024
3.91	−0.6950	−0.7190	24.9395	24.9595	49.8990
3.92	−0.7022	−0.7120	25.1903	25.2101	50.4004
$\frac{5}{4}\pi$	−0.7071	−0.7071	25.3672	25.3869	50.7540
3.93	−0.7092	−0.7050	25.4437	25.4633	50.9070
3.94	−0.7162	−0.6978	25.6996	25.7190	51.4186
3.95	−0.7232	−0.6906	25.9581	25.9773	51.9354
3.96	−0.7301	−0.6834	26.2191	26.2382	52.4573
3.97	−0.7369	−0.6760	26.4828	26.5017	52.9845
3.98	−0.7436	−0.6686	26.7492	26.7679	53.5170
3.99	−0.7502	−0.6612	27.0182	27.0367	54.0549
4.00	−0.7568	−0.6536	27.2899	27.3082	54.5982
4.10	−0.8183	−0.5748	30.1619	30.1784	60.3403
4.20	−0.8716	−0.4903	33.3357	33.3507	66.6863
4.30	−0.9162	−0.4008	36.8431	36.8567	73.6998
4.40	−0.9516	−0.3073	40.7193	40.7316	81.4509
4.50	−0.9775	−0.2108	45.0030	45.0141	90.0171
4.60	−0.9937	−0.1122	49.7371	49.7472	99.4843
4.70	−0.9999	−0.0124	54.9690	54.9781	109.9472
$\frac{3}{2}\pi$	−1	0	55.6544	55.6634	111.3178
4.80	−0.9962	0.0875	60.7511	60.7593	121.5104
4.90	−0.9824	0.1865	67.1412	67.1486	134.2898
5.00	−0.9589	0.2837	74.2032	74.2100	148.4132
5.10	−0.9258	0.3780	82.0079	82.0140	164.0219
5.20	−0.8834	0.4685	90.6334	90.6389	181.2722
5.30	−0.8323	0.5544	100.1659	100.1709	200.3368
5.40	−0.7728	0.6347	110.7010	110.7055	221.4064
$\frac{7}{4}\pi$	−0.7071	0.7071	122.0735	122.0776	244.1511
5.50	−0.7055	0.7087	122.3439	122.3480	244.6919
5.60	−0.6313	0.7756	135.2114	135.2150	270.4264
5.70	−0.5507	0.8347	149.4320	149.4354	298.8674
5.80	−0.4646	0.8855	165.1483	165.1513	330.2996
5.90	−0.3739	0.9275	182.5174	182.5201	365.0375
6.00	−0.2794	0.9602	201.7132	201.7156	403.4288
6.10	−0.1822	0.9833	222.9278	222.9300	445.8578
6.20	−0.0831	0.9965	246.3735	246.3755	492.7490
2π	0	1	267.7449	267.7468	535.4917
6.30	0.0168	0.9999	272.2850	272.2869	544.5719
6.40	0.1166	0.9932	300.9217	300.9434	601.8450

TABLE I

x	e^{-x}	A_x	B_x	C_x	D_x
3.85	0.0213	−0.0300	−0.0139	−0.0023	−0.0162
3.86	0.0211	−0.0297	−0.0139	−0.0020	−0.0159
3.87	0.0209	−0.0294	−0.0139	−0.0017	−0.0156
3.88	0.0206	−0.0292	−0.0139	−0.0014	−0.0153
3.89	0.0204	−0.0289	−0.0139	−0.0011	−0.0150
3.90	0.0202	−0.0286	−0.0140	−0.0008	−0.0147
3.91	0.0200	−0.0283	−0.0140	−0.0005	−0.0144
3.92	0.0198	−0.0280	−0.0140	−0.0002	−0.0141
$\frac{5}{4}\pi$	0.0197	−0.0278	−0.0140	0	−0.0139
3.93	0.0196	−0.0278	−0.0140	0.0001	−0.0139
3.94	0.0194	−0.0275	−0.0139	0.0003	−0.0136
3.95	0.0192	−0.0272	−0.0139	0.0005	−0.0133
3.96	0.0191	−0.0269	−0.0139	0.0008	−0.0130
3.97	0.0189	−0.0267	−0.0139	0.0011	−0.0128
3.98	0.0187	−0.0264	−0.0139	0.0014	−0.0125
3.99	0.0185	−0.0261	−0.0139	0.0017	−0.0122
4.00	0.0183	−0.0258	−0.0139	0.0019	−0.0120
4.10	0.0166	−0.0231	−0.0136	0.0040	−0.0096
4.20	0.0150	−0.0204	−0.0131	0.0057	−0.0074
4.30	0.0136	−0.0179	−0.0125	0.0070	−0.0055
4.40	0.0123	−0.0155	−0.0117	0.0079	−0.0038
4.50	0.0111	−0.0132	−0.0108	0.0085	−0.0023
4.60	0.0101	−0.0111	−0.0100	0.0089	−0.0012
4.70	0.0091	−0.0092	−0.0091	0.0090	−0.0001
$\frac{3}{2}\pi$	0.0090	−0.0090	−0.0090	0.0090	0
4.80	0.0082	−0.0075	−0.0082	0.0089	0.0007
4.90	0.0074	−0.0059	−0.0073	0.0087	0.0014
5.00	0.0067	−0.0046	−0.0065	0.0084	0.0019
5.10	0.0061	−0.0033	−0.0057	0.0080	0.0023
5.20	0.0055	−0.0023	−0.0049	0.0075	0.0026
5.30	0.0050	−0.0014	−0.0042	0.0069	0.0028
5.40	0.0045	−0.0006	−0.0035	0.0064	0.0029
$\frac{7}{4}\pi$	0.0041	0	−0.0029	0.0058	0.0029
5.50	0.0041	0.0000	−0.0029	0.0058	0.0029
5.60	0.0037	0.0005	−0.0023	0.0052	0.0029
5.70	0.0033	0.0009	−0.0018	0.0046	0.0028
5.80	0.0030	0.0013	−0.0014	0.0041	0.0027
5.90	0.0027	0.0015	−0.0010	0.0036	0.0026
6.00	0.0025	0.0017	−0.0007	0.0031	0.0024
6.10	0.0022	0.0018	−0.0004	0.0026	0.0022
6.20	0.0020	0.0019	−0.0002	0.0022	0.0020
2π	0.0019	0.0019	0	0.0019	0.0019
6.30	0.0018	0.0019	0.0001	0.0018	0.0019
6.40	0.0017	0.0018	0.0002	0.0015	0.0017

TABLE I

x	sin x	cos x	Sinh x	Cosh x	e^x
6.50	0.2151	0.9766	332.5701	332.5716	665.1416
6.60	0.3115	0.9502	367.5469	367.5483	735.0952
6.70	0.4048	0.9144	406.2023	406.2035	812.4058
6.80	0.4941	0.8694	448.9231	448.9242	897.8473
6.90	0.5784	0.8157	496.1369	496.1379	992.2747
7.00	0.6570	0.7539	548.3161	548.3170	1096.6332
$\frac{9}{4}\pi$	0.7071	0.7071	587.2412	587.2420	1174.4832
7.10	0.7290	0.6846	605.9831	605.9840	1211.9671
7.20	0.7937	0.6084	669.7150	669.7158	1339.4308
7.30	0.8504	0.5261	740.1496	740.1503	1480.2999
7.40	0.8987	0.4386	817.9919	817.9925	1635.9844
7.50	0.9380	0.3466	904.0209	904.0215	1808.0424
7.60	0.9679	0.2513	999.0977	999.0982	1998.1959
7.70	0.9882	0.1534	1104.1738	1104.1742	2208.3480
7.80	0.9985	0.0540	1220.3008	1220.3012	2440.6020
$\frac{5}{2}\pi$	1	0	1287.9850	1287.9854	2575.9705
7.90	0.9989	−0.0460	1348.6410	1348.6414	2697.2823
8.00	0.9894	−0.1455	1490.4788	1490.4792	2980.9580

TABLE I

e^{-x}	A_x	B_x	C_x	D_x
0.0015	0.0018	0.0003	0.0012	0.0018
0.0014	0.0017	0.0004	0.0009	0.0015
0.0012	0.0016	0.0005	0.0006	0.0013
0.0011	0.0015	0.0006	0.0004	0.0011
0.0010	0.0014	0.0006	0.0002	0.0009
0.0009	0.0013	0.0006	0.0001	0.0007
0.0009	0.0012	0.0006	0	0.0006
0.0008	0.0012	0.0006	−0.0000	0.0006
0.0007	0.0011	0.0006	−0.0001	0.0005
0.0007	0.0009	0.0006	−0.0002	0.0004
0.0006	0.0008	0.0006	−0.0003	0.0003
0.0006	0.0007	0.0005	−0.0003	0.0002
0.0005	0.0006	0.0005	−0.0004	0.0001
0.0005	0.0006	0.0005	−0.0004	0.0001
0.0004	0.0004	0.0004	−0.0004	0.0000
0.0004	0.0004	0.0004	−0.0004	0
0.0004	0.0004	0.0004	−0.0004	−0.0000
0.0003	0.0002	0.0003	−0.0004	−0.0001

TABLE II

E_I, E_{II}, F_I, F_{II}

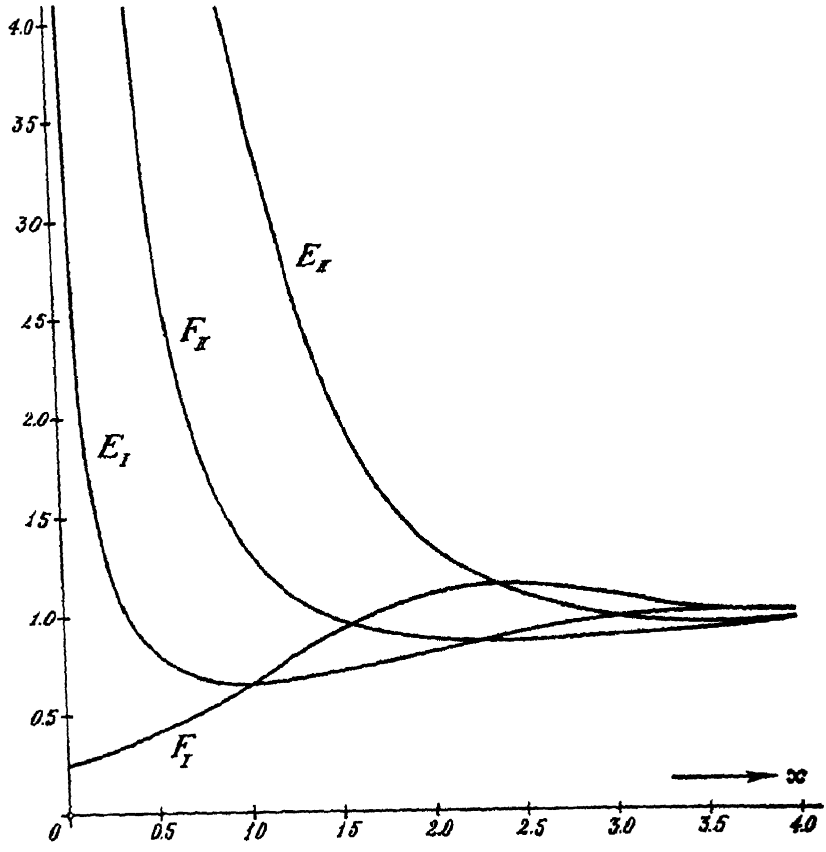

FIG. 163

$$E_I = \frac{1}{2}\frac{e^x}{\text{Sinh } x + \sin x}, \qquad F_I = \frac{1}{2}\frac{e^x}{\text{Cosh } x + \cos x},$$

$$E_{II} = \frac{1}{2}\frac{e^x}{\text{Sinh } x - \sin x}, \qquad F_{II} = \frac{1}{2}\frac{e^x}{\text{Cosh } x - \cos x}.$$

TABLE II

x	E_I	E_{II}	F_I	F_{II}
0	∞	∞	0.25000	∞
0.05	5.25320	25000.0	0.26281	215.51724
0.10	2.76342	1492.537	0.27629	55.15720
0.15	1.93656	505.050	0.29046	25.75992
0.20	1.52653	233.645	0.30533	15.26252
0.25	1.28391	123.916	0.32095	10.27432
0.30	1.12487	74.68260	0.33734	7.50694
0.35	1.01357	49.48046	0.35454	5.79408
0.40	0.93225	34.87966	0.37256	4.66135
0.45	0.87091	25.88662	0.39139	3.87342
0.50	0.82391	19.79414	0.41110	3.29728
0.55	0.78720	15.64211	0.43169	2.86311
0.60	0.75837	12.65662	0.45311	2.52915
0.65	0.73558	10.47449	0.47535	2.26588
0.70	0.71777	8.80282	0.49845	2.05339
0.75	0.70379	7.52615	0.52238	1.87995
0.80	0.69314	6.51466	0.54702	1.73702
0.85	0.68506	5.71690	0.57245	1.61687
0.90	0.67949	5.05817	0.59855	1.51538
0.95	0.67584	4.52018	0.62517	1.42945
1.00	0.67392	4.07398	0.65243	1.35512
1.05	0.67352	3.69904	0.67987	1.29189
1.10	0.67449	3.38066	0.70791	1.23609
1.15	0.67671	3.10810	0.73591	1.18813
1.20	0.67996	2.87455	0.76399	1.14611
1.25	0.68412	2.67294	0.79209	1.10904
1.30	0.68921	2.49669	0.81962	1.07704
1.35	0.69503	2.34307	0.84688	1.04865
1.40	0.70166	2.20658	0.87366	1.02353
1.45	0.70883	2.08690	0.89959	1.00158
1.50	0.71664	1.98016	0.92474	0.98213
1.55	0.72498	1.88484	0.94890	0.96502
1.60	0.73376	1.79973	0.97186	0.95006
1.65	0.74288	1.72378	0.99356	0.93696
1.70	0,75243	1.65500	1.01402	0.92544
1.75	0.76115	1.59795	1.03284	0.91558
1.80	0.77239	1.53685	1.05029	0.90701
1.85	0.78267	1.48615	1.06619	0.89967
1.90	0.79325	1.43962	1.08042	0.89352
.95	0.80376	1.39790	1.09309	0.88838

TABLE II

E_I	E_{II}	F_I	F_{II}
0.81454	1.35929	1.10409	0.88424
0.82506	1.32480	1.11387	0.88077
0.83584	1.29266	1.12208	0.87812
0.84643	1.26354	1.12871	0.87629
0.85710	1.23652	1.13394	0.87514
0.86738	1.21228	1.13829	0.87436
0.87754	1.18998	1.14097	0.87439
0.88755	1.16945	1.14299	0.87466
0.89737	1.15052	1.14360	0.87563
0.90699	1.13308	1.14358	0.87685
0.91604	1.11749	1.14267	0.87847
0.92515	1.10266	1.14090	0.88051
0.93364	1.08947	1.13880	0.88267
0.94179	1.07734	1.13586	0.88528
0.94978	1.06601	1.13260	0.88800
0.95720	1.04471	1.12881	0.89104
0.96423	1.04657	1.12473	0.89421
0.97083	1.03815	1.12038	0.89754
0.97716	1.03032	1.11578	0.90104
0.98303	1.02327	1.11118	0.90454
0.99828	1.00672	1.10634	0.90820
0.99823	1.00586	1.09646	0.91573
1.00650	0.99687	1.08665	0.92331
1.01313	0.98986	1.07717	0.93079
1.01844	0.98436	1.06779	0.93834
1.02261	0.98011	1.05897	0.94562
1.02559	0.97709	1.05070	0.95261
1.02755	0.97505	1.04318	0.95913
1.02870	0.97380	1.03616	0.96534
1.02924	0.97315	1.02986	0.97105
1.02895	0.97327	1.02423	0.97624
1.02824	0.97378	1.01930	0.98090
1.02714	0.97468	1.01480	0.98520
1.02583	0.97578	1.01094	0.98894
1.02412	0.97728	1.00750	0.99231
1.02220	0.97897	1.00450	0.99530
1.02051	0.98049	1.00230	0.99751
1.01862	0.98220	1.00012	0.99972
1.01674	0.98393	0.99853	1.00133
1.01488	0.98567	0.99715	1.00275
1.01322	0.98722	0.99616	1.00376

TABLE III

$$Z_1(x),\quad Z_2(x),\quad \frac{dZ_1(x)}{dx},\quad \frac{dZ_2(x)}{dx},$$

$$Z_3(x),\quad Z_4(x),\quad \frac{dZ_3(x)}{dx},\quad \frac{dZ_4(x)}{dx}.$$

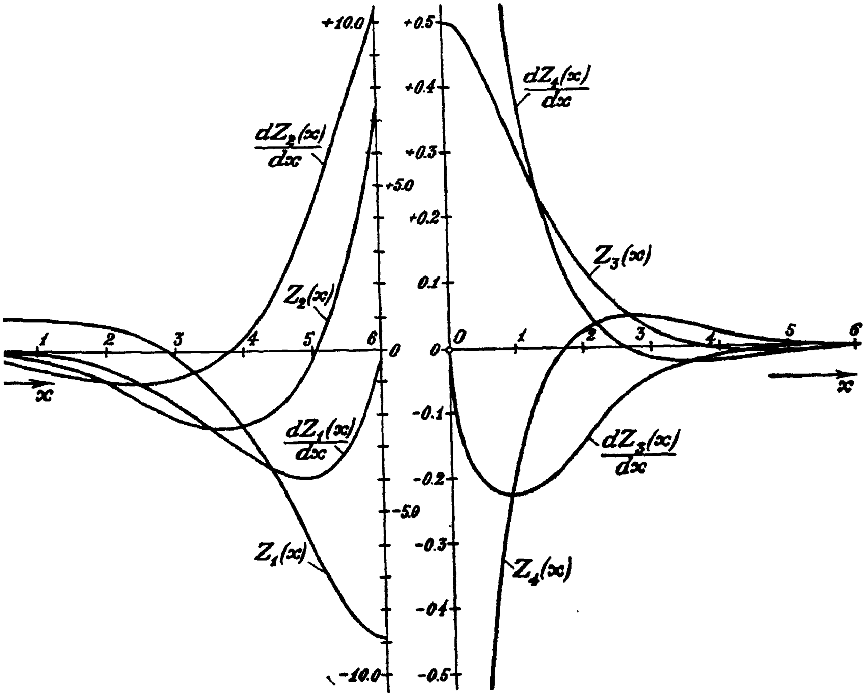

FIG. 164

$$= 1 - \frac{\left(\frac{x}{2}\right)^4}{2!^2} + \frac{\left(\frac{x}{2}\right)^8}{4!^2} - \frac{\left(\frac{x}{2}\right)^{12}}{6!^2} + \cdots, \qquad Z_3(x) = \frac{Z_1(x)}{2} - \frac{2}{\pi}\left[R_1 + \log_e \frac{\gamma x}{2}\cdot Z_2(x)\right],$$

$$= -\frac{\left(\frac{x}{2}\right)^2}{1!^2} + \frac{\left(\frac{x}{2}\right)^6}{3!^2} - \frac{\left(\frac{x}{2}\right)^{10}}{5!^2} + \cdots, \qquad Z_4(x) = \frac{Z_2(x)}{2} + \frac{2}{\pi}\left[R_2 + \log_e \frac{\gamma x}{2}\cdot Z_1(x)\right],$$

$$R_1 = \left(\frac{x}{2}\right)^2 - \frac{\varphi(3)}{3!^2}\left(\frac{x}{2}\right)^6 + \frac{\varphi(5)}{5!^2}\left(\frac{x}{2}\right)^{10} - \cdots,$$

$$R_2 = \frac{\varphi(2)}{2!^2}\left(\frac{x}{2}\right)^4 - \frac{\varphi(4)}{4!^2}\left(\frac{x}{2}\right)^8 + \frac{\varphi(6)}{6!^2}\left(\frac{x}{2}\right)^{12} - \cdots$$

$$\varphi(n) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} \quad \text{and} \quad \log_e \gamma = 0.577216.$$

TABLE III

x	$Z_1(x)$	$Z_2(x)$	$\frac{dZ_1(x)}{dx}$	$\frac{dZ_2(x)}{dx}$
0.00	1.0000	0.0000	0.0000	0.0000
0.01	1.0000	−0.0000	−0.0000	−0.0050
0.02	1.0000	−0.0001	−0.0000	−0.0100
0.03	1.0000	−0.0002	−0.0000	−0.0150
0.04	1.0000	−0.0004	−0.0000	−0.0200
0.05	1.0000	−0.0006	−0.0000	−0.0250
0.06	1.0000	−0.0009	−0.0000	−0.0300
0.07	1.0000	−0.0012	−0.0000	−0.0350
0.08	1.0000	−0.0016	−0.0000	−0.0400
0.09	1.0000	−0.0020	−0.0000	−0.0450
0.10	1.0000	−0.0025	−0.0001	−0.0500
0.11	1.0000	−0.0030	−0.0001	−0.0550
0.12	1.0000	−0.0036	−0.0001	−0.0600
0.13	1.0000	−0.0042	−0.0001	−0.0650
0.14	1.0000	−0.0049	−0.0002	−0.0700
0.15	1.0000	−0.0056	−0.0002	−0.0750
0.16	1.0000	−0.0064	−0.0003	−0.0800
0.17	1.0000	−0.0072	−0.0003	−0.0850
0.18	1.0000	−0.0081	−0.0004	−0.0900
0.19	1.0000	−0.0090	−0.0004	−0.0950
0.20	1.0000	−0.0100	−0.0005	−0.1000
0.21	1.0000	−0.0110	−0.0006	−0.1050
0.22	1.0000	−0.0121	−0.0007	−0.1100
0.23	1.0000	−0.0132	−0.0008	−0.1150
0.24	0.9999	−0.0144	−0.0009	−0.1200
0.25	0.9999	−0.0156	−0.0010	−0.1250
0.26	0.9999	−0.0169	−0.0011	−0.1300
0.27	0.9999	−0.0182	−0.0012	−0.1350
0.28	0.9999	−0.0196	−0.0014	−0.1400
0.29	0.9999	−0.0210	−0.0015	−0.1450
0.30	0.9999	−0.0225	−0.0017	−0.1500
0.31	0.9999	−0.0240	−0.0019	−0.1550
0.32	0.9998	−0.0256	−0.0020	−0.1600
0.33	0.9998	−0.0272	−0.0022	−0.1650
0.34	0.9998	−0.0289	−0.0025	−0.1700
0.35	0.9998	−0.0306	−0.0027	−0.1750
0.36	0.9997	−0.0324	−0.0029	−0.1800
0.37	0.9997	−0.0342	−0.0032	−0.1850
0.38	0.9997	−0.0361	−0.0034	−0.1900
0.39	0.9996	−0.0380	−0.0037	−0.1950

TABLE III

x	$Z_3(x)$	$Z_4(x)$	$\frac{dZ_3(x)}{dx}$	$\frac{dZ_4(x)}{dx}$
0.00	0.5000	$-\infty$	0.0000	$+\infty$
0.01	0.4999	−3.0056	−0.0166	63.6595
0.02	0.4997	−2.5643	−0.0288	31.8260
0.03	0.4993	−2.3063	−0.0394	21.2132
0.04	0.4989	−2.1232	−0.0488	15.9055
0.05	0.4984	−1.9813	−0.0575	12.7199
0.06	0.4978	−1.8653	−0.0655	10.5954
0.07	0.4971	−1.7674	−0.0730	9.0771
0.08	0.4963	−1.6825	−0.0800	7.9378
0.09	0.4955	−1.6078	−0.0866	7.0512
0.10	0.4946	−1.5409	−0.0929	6.3413
0.11	0.4936	−1.4805	−0.0989	5.7601
0.12	0.4926	−1.4254	−0.1046	5.2754
0.13	0.4915	−1.3748	−0.1100	4.8648
0.14	0.4904	−1.3279	−0.1152	4.5126
0.15	0.4892	−1.2843	−0.1201	4.2071
0.16	0.4880	−1.2436	−0.1248	3.9394
0.17	0.4867	−1.2054	−0.1294	3.7029
0.18	0.4854	−1.1695	−0.1337	3.4925
0.19	0.4840	−1.1355	−0.1379	3.3040
0.20	0.4826	−1.1034	−0.1419	3.1340
0.21	0.4812	−1.0728	−0.1458	2.9801
0.22	0.4797	−1.0437	−0.1495	2.8400
0.23	0.4782	−1.0160	−0.1531	2.7118
0.24	0.4767	−0.9894	−0.1565	2.5941
0.25	0.4751	−0.9640	−0.1598	2.4857
0.26	0.4735	−0.9397	−0.1630	2.3854
0.27	0.4718	−0.9163	−0.1661	2.2924
0.28	0.4701	−0.8938	−0.1690	2.2060
0.29	0.4684	−0.8722	−0.1719	2.1253
0.30	0.4667	−0.8513	−0.1746	2.0498
0.31	0.4650	−0.8312	−0.1773	1.9791
0.32	0.4632	−0.8117	−0.1798	1.9127
0.33	0.4614	−0.7929	−0.1823	1.8502
0.34	0.4595	−0.7747	−0.1846	1.7912
0.35	0.4577	−0.7571	−0.1869	1.7355
0.36	0.4558	−0.7400	−0.1891	1.6828
0.37	0.4539	−0.7234	−0.1912	1.6329
0.38	0.4520	−0.7073	−0.1932	1.5854
0.39	0.4500	−0.6917	−0.1952	1.5403

TABLE III

x	$Z_1(x)$	$Z_2(x)$	$\frac{dZ_1(x)}{dx}$	$\frac{dZ_2(x)}{dx}$
0.40	0.9996	−0.0400	−0.0040	−0.2000
0.41	0.9996	−0.0420	−0.0043	−0.2050
0.42	0.9995	−0.0441	−0.0046	−0.2100
0.43	0.9995	−0.0462	−0.0050	−0.2150
0.44	0.9994	−0.0484	−0.0053	−0.2200
0.45	0.9994	−0.0506	−0.0057	−0.2250
0.46	0.9993	−0.0529	−0.0061	−0.2299
0.47	0.9992	−0.0552	−0.0065	−0.2349
0.48	0.9992	−0.0576	−0.0069	−0.2399
0.49	0.9991	−0.0600	−0.0074	−0.2449
0.50	0.9990	−0.0625	−0.0078	−0.2499
0.51	0.9989	−0.0650	−0.0083	−0.2549
0.52	0.9989	−0.0676	−0.0088	−0.2599
0.53	0.9988	−0.0702	−0.0093	−0.2649
0.54	0.9987	−0.0729	−0.0098	−0.2699
0.55	0.9986	−0.0756	−0.0104	−0.2749
0.56	0.9985	−0.0784	−0.0110	−0.2799
0.57	0.9984	−0.0812	−0.0116	−0.2848
0.58	0.9982	−0.0841	−0.0122	−0.2898
0.59	0.9981	−0.0870	−0.0128	−0.2948
0.60	0.9980	−0.0900	−0.0135	−0.2998
0.61	0.9978	−0.0930	−0.0142	−0.3048
0.62	0.9977	−0.0961	−0.0149	−0.3098
0.63	0.9975	−0.0992	−0.0156	−0.3147
0.64	0.9974	−0.1024	−0.0164	−0.3197
0.65	0.9972	−0.1056	−0.0172	−0.3247
0.66	0.9970	−0.1089	−0.0180	−0.3297
0.67	0.9969	−0.1122	−0.0188	−0.3346
0.68	0.9967	−0.1156	−0.0196	−0.3396
0.69	0.9965	−0.1190	−0.0205	−0.3446
0.70	0.9962	−0.1224	−0.0214	−0.3496
0.71	0.9960	−0.1260	−0.0224	−0.3545
0.72	0.9958	−0.1295	−0.0233	−0.3595
0.73	0.9956	−0.1332	−0.0243	−0.3645
0.74	0.9953	−0.1368	−0.0253	−0.3694
0.75	0.9951	−0.1405	−0.0264	−0.3744
0.76	0.9948	−0.1443	−0.0274	−0.3793
0.77	0.9945	−0.1481	−0.0285	−0.3843
0.78	0.9942	−0.1520	−0.0296	−0.3892
0.79	0.9939	−0.1559	−0.0308	−0.3942

TABLE III

x	$Z_3(x)$	$Z_4(x)$	$\frac{dZ_3(x)}{dx}$	$\frac{dZ_4(x)}{dx}$
0.40	0.4480	−0.6765	−0.1970	1.4974
0.41	0.4461	−0.6617	−0.1988	1.4564
0.42	0.4441	−0.6474	−0.2006	1.4174
0.43	0.4421	−0.6334	−0.2022	1.3800
0.44	0.4400	−0.6198	−0.2038	1.3443
0.45	0.4380	−0.6065	−0.2054	1.3101
0.46	0.4359	−0.5935	−0.2068	1.2773
0.47	0.4338	−0.5809	−0.2082	1.2458
0.48	0.4318	−0.5686	−0.2096	1.2156
0.49	0.4297	−0.5566	−0.2109	1.1865
0.50	0.4275	−0.5449	−0.2121	1.1585
0.51	0.4254	−0.5334	−0.2133	1.1316
0.52	0.4233	−0.5222	−0.2144	1.1056
0.53	0.4211	−0.5113	−0.2155	1.0806
0.54	0.4190	−0.5006	−0.2165	1.0564
0.55	0.4168	−0.4902	−0.2175	1.0330
0.56	0.4146	−0.4800	−0.2184	1.0105
0.57	0.4124	−0.4700	−0.2193	0.9887
0.58	0.4102	−0.4602	−0.2201	0.9675
0.59	0.4080	−0.4506	−0.2209	0.9471
0.60	0.4058	−0.4413	−0.2216	0.9273
0.61	0.4036	−0.4321	−0.2224	0.9080
0.62	0.4014	−0.4231	−0.2230	0.8894
0.63	0.3991	−0.4143	−0.2236	0.8713
0.64	0.3969	−0.4057	−0.2242	0.8538
0.65	0.3946	−0.3972	−0.2247	0.8367
0.66	0.3924	−0.3889	−0.2252	0.8201
0.67	0.3902	−0.3808	−0.2257	0.8040
0.68	0.3879	−0.3728	−0.2261	0.7883
0.69	0.3856	−0.3650	−0.2265	0.7730
0.70	0.3834	−0.3574	−0.2268	0.7582
0.71	0.3811	−0.3499	−0.2272	0.7437
0.72	0.3788	−0.3425	−0.2274	0.7296
0.73	0.3765	−0.3353	−0.2277	0.7159
0.74	0.3743	−0.3282	−0.2279	0.7024
0.75	0.3720	−0.3212	−0.2281	0.6894
0.76	0.3697	−0.3144	−0.2282	0.6766
0.77	0.3674	−0.3077	−0.2284	0.6642
0.78	0.3651	−0.3011	−0.2285	0.6520
0.79	0.3628	−0.2947	−0.2285	0.6402

TABLE III

x	$Z_1(x)$	$Z_2(x)$	$\dfrac{dZ_1(x)}{dx}$	$\dfrac{dZ_2(x)}{dx}$
0.80	0.9936	−0.1599	−0.0320	−0.3991
0.81	0.9933	−0.1639	−0.0332	−0.4041
0.82	0.9929	−0.1680	−0.0344	−0.4090
0.83	0.9926	−0.1721	−0.0357	−0.4140
0.84	0.9922	−0.1762	−0.0370	−0.4189
0.85	0.9918	−0.1805	−0.0384	−0.4238
0.86	0.9915	−0.1847	−0.0397	−0.4288
0.87	0.9910	−0.1890	−0.0411	−0.4337
0.88	0.9906	−0.1934	−0.0426	−0.4386
0.89	0.9902	−0.1978	−0.0440	−0.4435
0.90	0.9898	−0.2023	−0.0455	−0.4485
0.91	0.9893	−0.2068	−0.0471	−0.4534
0.92	0.9888	−0.2113	−0.0486	−0.4583
0.93	0.9883	−0.2159	−0.0502	−0.4632
0.94	0.9878	−0.2206	−0.0519	−0.4681
0.95	0.9873	−0.2253	−0.0535	−0.4730
0.96	0.9867	−0.2301	−0.0553	−0.4779
0.97	0.9862	−0.2349	−0.0570	−0.4828
0.98	0.9856	−0.2397	−0.0588	−0.4876
0.99	0.9850	−0.2446	−0.0606	−0.4925
1.00	0.9844	−0.2496	−0.0624	−0.4974
1.02	0.9831	−0.2596	−0.0663	−0.5071
1.04	0.9817	−0.2699	−0.0702	−0.5168
1.06	0.9803	−0.2803	−0.0744	−0.5265
1.08	0.9788	−0.2909	−0.0786	−0.5362
1.10	0.9771	−0.3017	−0.0831	−0.5458
1.12	0.9754	−0.3127	−0.0877	−0.5554
1.14	0.9736	−0.3239	−0.0925	−0.5650
1.16	0.9717	−0.3353	−0.0974	−0.5745
1.18	0.9697	−0.3469	−0.1025	−0.5840
1.20	0.9676	−0.3587	−0.1078	−0.5935
1.22	0.9654	−0.3707	−0.1133	−0.6030
1.24	0.9631	−0.3828	−0.1189	−0.6124
1.26	0.9607	−0.3952	−0.1248	−0.6217
1.28	0.9581	−0.4077	−0.1308	−0.6310
1.30	0.9554	−0.4204	−0.1370	−0.6403
1.32	0.9526	−0.4333	−0.1434	−0.6496
1.34	0.9497	−0.4464	−0.1500	−0.6588
1.36	0.9466	−0.4597	−0.1567	−0.6679
1.38	0.9434	−0.4731	−0.1637	−0.6770

TABLE III

x	$Z_3(x)$	$Z_4(x)$	$\frac{dZ_3(x)}{dx}$	$\frac{dZ_4(x)}{dx}$
0.80	0.3606	−0.2883	−0.2286	0.6286
0.81	0.3583	−0.2821	−0.2286	0.6172
0.82	0.3560	−0.2760	−0.2286	0.6062
0.83	0.3537	−0.2700	−0.2285	0.5953
0.84	0.3514	−0.2641	−0.2285	0.5847
0.85	0.3491	−0.2583	−0.2284	0.5744
0.86	0.3468	−0.2526	−0.2283	0.5642
0.87	0.3446	−0.2470	−0.2281	0.5543
0.88	0.3423	−0.2415	−0.2280	0.5446
0.89	0.3400	−0.2361	−0.2278	0.5351
0.90	0.3377	−0.2308	−0.2276	0.5258
0.91	0.3355	−0.2256	−0.2273	0.5166
0.92	0.3332	−0.2204	−0.2271	0.5077
0.93	0.3309	−0.2154	−0.2268	0.4989
0.94	0.3286	−0.2105	−0.2265	0.4904
0.95	0.3264	−0.2056	−0.2262	0.4819
0.96	0.3241	−0.2008	−0.2258	0.4737
0.97	0.3219	−0.1961	−0.2255	0.4656
0.98	0.3196	−0.1915	−0.2251	0.4576
0.99	0.3174	−0.1870	−0.2247	0.4498
1.00	0.3151	−0.1825	−0.2243	0.4422
1.02	0.3106	−0.1738	−0.2234	0.4273
1.04	0.3062	−0.1654	−0.2225	0.4130
1.06	0.3018	−0.1573	−0.2215	0.3992
1.08	0.2973	−0.1495	−0.2204	0.3859
1.10	0.2929	−0.1419	−0.2193	0.3730
1.12	0.2886	−0.1345	−0.2182	0.3606
1.14	0.2842	−0.1274	−0.2169	0.3486
1.16	0.2799	−0.1206	−0.2156	0.3370
1.18	0.2756	−0.1140	−0.2143	0.3258
1.20	0.2713	−0.1076	−0.2129	0.3149
1.22	0.2671	−0.1014	−0.2115	0.3044
1.24	0.2628	−0.0954	−0.2100	0.2942
1.26	0.2587	−0.0896	−0.2086	0.2844
1.28	0.2545	−0.0840	−0.2070	0.2748
1.30	0.2504	−0.0786	−0.2054	0.2656
1.32	0.2463	−0.0734	−0.2038	0.2567
1.34	0.2422	−0.0683	−0.2022	0.2480
1.36	0.2382	−0.0634	−0.2005	0.2396
1.38	0.2342	−0.0587	−0.1988	0.2314

TABLE III

x	$Z_1(x)$	$Z_2(x)$	$\frac{dZ_1(x)}{dx}$	$\frac{dZ_2(x)}{dx}$
1.40	0.9401	−0.4867	−0.1709	−0.6860
1.42	0.9366	−0.5005	−0.1783	−0.6950
1.44	0.9329	−0.5145	−0.1859	−0.7039
1.46	0.9291	−0.5287	−0.1937	−0.7127
1.48	0.9252	−0.5430	−0.2018	−0.7215
1.50	0.9211	−0.5576	−0.2100	−0.7302
1.52	0.9168	−0.5723	−0.2185	−0.7389
1.54	0.9123	−0.5871	−0.2272	−0.7475
1.56	0.9077	−0.6022	−0.2361	−0.7560
1.58	0.9029	−0.6174	−0.2452	−0.7644
1.60	0.8979	−0.6327	−0.2545	−0.7727
1.62	0.8927	−0.6483	−0.2641	−0.7810
1.64	0.8873	−0.6640	−0.2740	−0.7892
1.66	0.8817	−0.6798	−0.2840	−0.7972
1.68	0.8760	−0.6959	−0.2943	−0.8052
1.70	0.8700	−0.7120	−0.3048	−0.8131
1.72	0.8638	−0.7284	−0.3156	−0.8209
1.74	0.8573	−0.7449	−0.3266	−0.8286
1.76	0.8507	−0.7615	−0.3379	−0.8361
1.78	0.8438	−0.7783	−0.3494	−0.8436
1.80	0.8367	−0.7953	−0.3612	−0.8509
1.82	0.8294	−0.8124	−0.3732	−0.8581
1.84	0.8218	−0.8296	−0.3855	−0.8652
1.86	0.8140	−0.8470	−0.3980	−0.8722
1.88	0.8059	−0.8645	−0.4108	−0.8790
1.90	0.7975	−0.8821	−0.4238	+0.8857
1.92	0.7889	−0.8999	−0.4372	−0.8923
1.94	0.7800	−0.9178	−0.4507	−0.8987
1.96	0.7709	−0.9358	−0.4646	−0.9050
1.98	0.7614	−0.9540	−0.4787	−0.9111
2.0	0.7517	−0.9723	−0.4931	−0.9170
2.1	0.6987	−1.0654	−0.5690	−0.9442
2.2	0.6377	−1.1610	−0.6520	−0.9666
2.3	0.5680	−1.2585	−0.7420	−0.9836
2.4	0.4890	−1.3575	−0.8392	−0.9944
2.5	0.4000	−1.4572	−0.9436	−0.9983
2.6	0.3001	−1.5569	−1.0552	−0.9943
2.7	0.1887	−1.6557	−1.1737	−0.9815
2.8	0.0651	−1.7529	−1.2993	−0.9589
2.9	−0.0714	−1.8472	−1.4315	−0.9256

TABLE III

x	$Z_3(x)$	$Z_4(x)$	$\frac{dZ_3(x)}{dx}$	$\frac{dZ_4(x)}{dx}$
1.40	0.2302	−0.0542	−0.1971	0.2235
1.42	0.2263	−0.0498	−0.1954	0.2158
1.44	0.2224	−0.0456	−0.1936	0.2084
1.46	0.2186	−0.0416	−0.1918	0.2011
1.48	0.2148	−0.0375	−0.1900	0.1941
1.50	0.2110	−0.0337	−0.1882	0.1873
1.52	0.2072	−0.0300	−0.1864	0.1807
1.54	0.2035	−0.0265	−0.1845	0.1742
1.56	0.1999	−0.0230	−0.1826	0.1680
1.58	0.1962	−0.0198	−0.1807	0.1619
1.60	0.1926	−0.0166	−0.1788	0.1560
1.62	0.1891	−0.0135	−0.1769	0.1503
1.64	0.1855	−0.0106	−0.1750	0.1448
1.66	0.1821	−0.0077	−0.1731	0.1394
1.68	0.1786	−0.0050	−0.1711	0.1341
1.70	0.1752	−0.0024	−0.1692	0.1290
1.72	0.1718	0.0002	−0.1672	0.1241
1.74	0.1685	0.0026	−0.1653	0.1193
1.76	0.1652	0.0050	−0.1634	0.1146
1.78	0.1620	0.0072	−0.1614	0.1100
1.80	0.1588	0.0094	−0.1594	0.1056
1.82	0.1556	0.0114	−0.1575	0.1014
1.84	0.1525	0.0134	−0.1555	0.0972
1.86	0.1494	0.0153	−0.1536	0.0931
1.88	0.1463	0.0171	−0.1516	0.0892
1.90	0.1433	0.0189	−0.1496	0.0854
1.92	0.1404	0.0206	−0.1477	0.0817
1.94	0.1374	0.0222	−0.1458	0.0781
1.96	0.1345	0.0237	−0.1438	0.0746
1.98	0.1317	0.0251	−0.1419	0.0712
2.0	0.1289	0.0265	−0.1399	0.0679
2.1	0.1153	0.0325	−0.1304	0.0527
2.2	0.1028	0.0371	−0.1210	0.0397
2.3	0.0911	0.0405	−0.1120	0.0285
2.4	0.0804	0.0429	−0.1032	0.0189
2.5	0.0705	0.0444	−0.0948	0.0108
2.6	0.0614	0.0451	−0.0868	0.0039
2.7	0.0531	0.0452	−0.0791	−0.0018
2.8	0.0455	0.0447	−0.0719	−0.0066
2.9	0.0387	0.0439	−0.0650	−0.0105

TABLE III

x	$Z_1(x)$	$Z_2(x)$	$\frac{dZ_1(x)}{dx}$	$\frac{dZ_2(x)}{dx}$
3.0	−0.2214	−1.9376	−1.5698	−0.8804
3.1	−0.3855	−2.0228	−1.7141	−0.8223
3.2	−0.5644	−2.1016	−1.8636	−0.7499
3.3	−0.7584	−2.1723	−2.0177	−0.6621
3.4	−0.9680	−2.2334	−2.1755	−0.5577
3.5	−1.1936	−2.2832	−2.3361	−0.4353
3.6	−1.4353	−2.3199	−2.4983	−0.2936
3.7	−1.6933	−2.3413	−2.6608	−0.1315
3.8	−1.9674	−2.3454	−2.8221	0.0526
3.9	−2.2576	−2.3300	−2.9808	0.2596
4.0	−2.5634	−2.2927	−3.1346	0.4912
4.1	−2.8843	−2.2309	−3.2819	0.7482
4.2	−3.2195	−2.1422	−3.4199	1.0318
4.3	−3.5679	−2.0236	−3.5465	1.3433
4.4	−3.9283	−1.8726	−3.6587	1.6833
4.5	−4.2991	−1.6860	−3.7536	2.0526
4.6	−4.6784	−1.4610	−3.8280	2.4520
4.7	−5.0639	−1.1946	−3.8782	2.8818
4.8	−5.4531	−0.8837	−3.9006	3.3422
4.9	−5.8429	−0.5251	−3.8910	3.8330
5.0	−6.2301	−0.1160	−3.8454	4.3542
5.1	−6.6107	0.3467	−3.7589	4.9046
5.2	−6.9803	0.8658	−3.6270	5.4835
5.3	−7.3344	1.4443	−3.4446	6.0893
5.4	−7.6674	2.0845	−3.2063	6.7198
5.5	−7.9736	2.7890	−2.9070	7.3729
5.6	−8.2466	3.5597	−2.5409	8.0453
5.7	−8.4794	4.3986	−2.1024	8.7336
5.8	−8.6644	5.3068	−1.5856	9.4332
5.9	−8.7937	6.2854	−0.9844	10.1394
6.0	−8.8583	7.3347	−0.2931	10.8462

TABLE III

x	$Z_3(x)$	$Z_4(x)$	$\frac{dZ_3(x)}{dx}$	$\frac{dZ_4(x)}{dx}$
3.0	0.0326	0.0427	−0.0586	−0.0137
3.1	0.0270	0.0412	−0.0526	−0.0162
3.2	0.0220	0.0394	−0.0470	−0.0180
3.3	0.0176	0.0376	−0.0418	−0.0195
3.4	0.0137	0.0356	−0.0369	−0.0204
3.5	0.0102	0.0335	−0.0325	−0.0210
3.6	0.0072	0.0314	−0.0284	−0.0213
3.7	0.0045	0.0293	−0.0246	−0.0213
3.8	0.0022	0.0272	−0.0212	−0.0210
3.9	0.0003	0.0251	−0.0180	−0.0206
4.0	−0.0014	0.0230	−0.0152	−0.0200
4.1	−0.0028	0.0211	−0.0127	−0.0193
4.2	−0.0039	0.0192	−0.0104	−0.0185
4.3	−0.0049	0.0174	−0.0083	−0.0177
4.4	−0.0056	0.0156	−0.0065	−0.0168
4.5	−0.0062	0.0140	−0.0049	−0.0158
4.6	−0.0066	0.0125	−0.0035	−0.0148
4.7	−0.0069	0.0110	−0.0023	−0.0138
4.8	−0.0071	0.0097	−0.0012	−0.0129
4.9	−0.0071	0.0085	−0.0003	−0.0119
5.0	−0.0071	0.0073	0.0005	−0.0109
5.1	−0.0070	0.0063	0.0012	−0.0100
5.2	−0.0069	0.0053	0.0017	−0.0091
5.3	−0.0067	0.0045	0.0022	−0.0083
5.4	−0.0065	0.0037	0.0025	−0.0075
5.5	−0.0062	0.0029	0.0028	−0.0067
5.6	−0.0059	0.0023	0.0030	−0.0060
5.7	−0.0056	0.0017	0.0032	−0.0053
5.8	−0.0053	0.0012	0.0033	−0.0047
5.9	−0.0049	0.0008	0.0033	−0.0041
6.0	−0.0046	0.0004	0.0033	−0.0036